11-1993

Parallel Max Cut Approximations

Grammati E. Pantziou
Dartmouth College

Paul G. Spirakis
Patras University

Christos D. Zaroliagis
Computer Technology Institute, Patras, Greece

PARALLEL MAX CUT APPROXIMATIONS

Grammati E. Pantziou
Paul G. Spirakis
Christos D. Zaroliagis

Technical Report PCS-TR93-201

11/93

Parallel Max Cut Approximations*

Grammati E. Pantziou [1,2] Paul G. Spirakis [1,3] Christos D. Zaroliagis [1]

(1) Computer Technology Institute, P.O. Box 1122, 26110 Patras, Greece

(2) Department of Mathematics and Computer Science, Dartmouth College, Hanover NH 03755, USA

(3) Department of Computer Science and Engineering, Patras University, 26500 Patras, Greece

Abstract

Given a graph with positive integer edge weights one may ask whether there exists an edge cut whose weight is bigger than a given number. This problem is NP-complete. We present here an approximation algorithm in NC which provides tight upper bounds to the proportion of edge cuts whose size is bigger than a given number. Our technique is based on the methods to convert randomized parallel algorithms into deterministic ones introduced by Karp and Wigderson. The basic idea of those methods is to replace an exponentially large sample space by one of polynomial size. In this work, we prove the interesting result that the statistical distance of random variables of the small sample space is bigger than the statistical distance of corresponding variables of the exponentially large space, which is the space of all edge cuts taken equiprobably.

Keywords: Parallel algorithms, approximation algorithm, derandomization method, variance, Chebyshev's inequality.

1 Introduction

Given a graph $G = (V, E)$, where $n = |V|$ and $m = |E|$, with integer weights $w(e) > 0$ for each $e \in E$ and a positive integer K, the max cut problem asks whether there is a partition of V into disjoint sets V_1 and V_2, such that the sum of the weights of the edges that have one endpoint in V_1 and the other in V_2 is at least K. The problem is NP-complete [4] and remains so, even when $w(e) = 1$ for all $e \in E$ (and even when the maximum degree in G is less than or equal to 3) [5]. There are 2^n edge cuts (including the cases whether $V_1 = \emptyset$ or $V_2 = \emptyset$). It is clear that the variation of the problem which asks whether the proportion of edge cuts whose weight is at least K is bigger than a given rational number, is still NP-hard and in fact #P-complete (since it is the enumeration version of an NP-complete problem [4]).

*This work was partially supported by the EEC ESPRIT Basic Research Action No. 7141 (ALCOM II), by the Greek Ministry of Education and by NSF grant CDA-9211155.

The max cut problem and its approximate version are central in Computer-Aided Design (CAD) for Very Large Scale Integration (VLSI) layout. It is listed in [12] as one of the most important and applicable problems in Physical Design Automation. For example, all of the existing methodologies for the Two-Layer Constraint Via Minimization (2-CVM) problem are based on solving the max cut problem on a non-necessarily planar graph (which is called the cluster graph). Furhtermore, it has been shown that unless the cluster graph is planar, the 2-CVM problem is NP-complete [12]. Thus, the approximate max cut problem is of particular importance in the more general case of non-planar cluster graphs. An additional application of the max cut problem is shown in [8] where the authors present a heuristic approach for the min cut problem based on efficient solutions to the max cut problem which, unfortunately, is polynomially solvable only for planar graphs. Thus, the technique presented in [8] is applicable only for special cases where the auxiliary graph that the authors construct is planar. Therefore, efficient approximate solutions to the max cut problem can make the method in [8] more applicable.

One may consider the space of all possible edge cuts of a weighted graph (2^n of them). Thus, by forming partitions of the vertex set at random, one may construct fast Monte Carlo algorithms to approximate the max cut problem. Such an approach is, in general, not very successful unless many random partitions are formed and running averages are estimated.

We prove here the surprising result that there is an $\Theta(n)$-sized subset of edge cuts which has the same mean edge cut weight and *bigger* variance compared with the space of all cuts, taken equiprobably. This set of cuts can be constructed in NC, as well as the mean value and the variance can be computed, in $O(\log n)$ parallel time by using $O(n^2 + nm/\log n)$ processors on an EREW PRAM. Furthermore, we prove that we can use the variance of the smaller space to apply Chebyshev's inequality on the random variable of the large space. Since the variance of the smaller space is bigger than that of the space of all cuts, we get a better upper bound on the fraction of edge cuts whose weight is bigger than a given number, than we would have gotten with the smaller variance. Thus, this single, easy to construct, set eliminates the need for the Monte Carlo approach and directly provides a tight upper bound on the number (or the fraction) of edge cuts whose weight is bigger than a given number. (This upper bound is tight in the sense that Chebyshev's inequality cannot be made more precise from just the mean and the variance of an otherwise arbitrary distribution.)

Our approach is based on the techniques for removing randomness from parallel algorithms, introduced by Karp and Wigderson [7] and analyzed in depth by Luby [9, 10], Goldberg and Spencer [6], and Alon, Babai and Itai [1]. These ideas exploit the fact that 2-

wise (or d-wise) independence (instead of complete independence) is all one needs in certain randomized algorithms, and this remark, in turn, means that exponentially smaller sample spaces suffice compared to what is required for full independence. We were particularly motivated by the so-called simple PROFIT/COST problem of Luby in [10]. In that work, Luby examined the expected value of a "benefit" function and proved that it stays the same in the smaller space. We examine, in addition, the variance and prove that it increases in the smaller space. Thus the smaller space is a good predictor of extreme values in the large space.

2 Preliminaries

Definition 1 *Let $C(V_1, V_2)$ be the set of edges with one endpoint in V_1 and the other in V_2. Let $W(C)$ be the sum of $w(e)$ for $e \in C(V_1, V_2)$.*

There is an one-to-one correspondence between each partition (V_1, V_2) of V and each 0/1 labelling $\hat{l} = \{l_i \in \{0,1\} : i \in V\}$ in the obvious way (the vertices of V_1 are 0-labelled). Hence, each 0/1 labelling $\hat{l}$ of the graph G defines an edge cut $C(\hat{l})$. Consider a probability distribution on labellings $\hat{l}$ such that each label l_v is independently chosen to be 1 or 0 with probability 1/2. In this space, each of the 2^n labellings is equally likely. Let Ω_1 be the corresponding sample space, and (Ω_1, Pr_1) the probability space.

Definition 2 *Let $weight(v,w) = w(e)$ if $e = \{v,w\} \in E$ and $weight(v,w) = 0$ otherwise. Let a function $COST : V^2 \times \{0,1\}^2 \to Z^+ \cup \{0\}$ be defined such that, for any two vertices v,w and their labels l_v, l_w in a labelling $\hat{l}$: $COST(v,w,l_v,l_w) = weight(v,w)$ if $l_v \neq l_w$, otherwise $COST(v,w,l_v,l_w) = 0$.*

For ease of notation, we will write $COST(l_v, l_w)$ instead of $COST(v,w,l_v,l_w)$. Clearly then, the weight $W(C)$ of a cut $C(\hat{l})$ is $W(C(\hat{l})) = \sum_{\{v,w\}\in E} COST(l_v, l_w)$ where $l_v, l_w \in \hat{l}$.

Let us define the random variable (r.v.) $\widetilde{W}$ on Ω_1 which is the weight of the edge cut corresponding to a randomly chosen labelling $\hat{l}$ according to the probability distribution defined on Ω_1. In fact, $\widetilde{W}$ is a function of the n independent r.v.'s l_v, $v \in V$. Let $\overline{W}$ and σ^2 be the mean value and variance of the r.v. $\widetilde{W}$. By Chebyshev's inequality then (for any $t > 0$)

$$Pr(\ |\widetilde{W} - \overline{W}| \geq t \cdot \sigma\ ) \leq \frac{1}{t^2} \quad (1)$$

i.e. by knowing the mean value of the weight of edge cuts and the corresponding variance, we get an upper bound on the fraction of edge cuts whose weight is greater than $t \cdot \sigma$. Clearly (1) holds for $\widetilde{W} =$ *the weight of the maximum cut*, too.

3 Properties of the $\Theta(n)$-sized Sample Space

Assume now that the r.v.'s l_i ($i \in V$) are defined (as in [10]) on a specially designed probability space (Ω_2, Pr_2), containing only $\Theta(n)$ sample points ω, such that the l_i are only *pairwise* independent. Then we can compute efficiently in parallel (in NC) the weights of the cuts corresponding to all the sample points $\omega \in \Omega_2$, and compute (in NC) the corresponding mean values, variances and other statistics of the weights. This computation will be meaningful only when the statistics in (Ω_2, Pr_2) can be used to derive statistics for the space Ω_1 (and the corresponding distribution). We use here the (Ω_2, Pr_2) as constructed by Luby in [10]; namely, $\Omega_2 = \{0,1\}^{k+1}, k = \lceil \log n \rceil$. For each $\omega \in \Omega_2$ we have $Pr(\omega) = 2^{-(k+1)}$. The r.v.'s $\hat{l} = \{l_i \in \{0,1\} : i \in V\}$ are defined on (Ω_2, Pr_2) as follows: For all $i \in V$ let $< i_1, i_2, \ldots, i_k >$ be the binary expansion of i. For all $\omega \in \Omega_2$, define l_i at ω to be: $l_i(\omega) = (\sum_{j=1}^{k}(i_j \cdot \omega_j) + \omega_{k+1}) \bmod 2$.

With this definition the r.v.'s are uniformly distributed and pairwise independent. Furthermore, each label l_i is equally likely to be 0 or 1. Luby publicized in his talks that the r.v.'s are also three-wise independent. The following fact has also been proved independently in [2, 11].

Fact 1 *The random variables l_i defined above are three-wise independent.*

Note that the r.v.'s l_i are not four-wise independent (see also [2]), since the size of a four-wise independent probability space is $\Omega(n^2)$. The following fact can be readily obtained from [10] (see also [11]).

Fact 2 *The mean value $\overline{W_2}$ of the weights of edge cuts in the space (Ω_2, Pr_2) is the same as in the space (Ω_1, Pr_1) and is given by the formula:*

$$\overline{W_2} = E[\sum_{\{v,w\} \in E} COST(l_v, l_w)] = \sum_{\{v,w\} \in E} E[COST(l_v, l_w)] = \frac{1}{2} \sum_{e \in E} w(e) \quad (2)$$

where $e = \{v, w\}$ and $E[\,]$ is the expected value operator.

Before we proceed to our main result, we first prove the following technical lemma.

Lemma 1 *Let U_1, U_2, L_1, L_2 be events in space (Ω_2, Pr_2) defined as follows: U_1 = "$l_v = 0 \wedge l_w = 1$", U_2 = "$l_v = 1 \wedge l_w = 0$", L_1 = "$l_u = 0 \wedge l_z = 1$", L_2 = "$l_u = 1 \wedge l_z = 0$", where v, w, u, z are all distinct. Then, $\frac{1}{4} \leq Pr(U_i | L_j) \leq \frac{1}{2}$ for $i = 1, 2$ and $j = 1, 2$.*

Proof: Let the $n \times 2n$ matrix A_k of $0, 1$ be the matrix such that $A_k(i, \omega) = l_i(\omega)$ ($n = 2^k$). As it can be easily checked (from [10]) A_{k+1} is constructed from A_k as it is shown in figure 1. Note that $A_k^I(i,\omega) = A_k^{II}(i,\omega) = A_k^{III}(i,\omega) = A_k(i,\omega)$ and $\overline{A_k}(i,\omega) = \overline{A_k(i,\omega)}\ \forall i, \omega$. (The

superscript denotes the different occurrences of A_k while the bar denotes complementation.) This can be verified by construction of $l_i(\omega)$. In the sequel with Ω_k, Ω_{k+1} we shall denote the corresponding sample spaces of A_k, A_{k+1} respectively. We now prove the lemma by induction for any $n \geq 4$. We prove it here only for $i = 1, j = 1$. The other cases are similar.

For $n = 4$ it can be easily verified that the assertion of the lemma holds. Suppose that it holds for $n = 2^k$. Again for the sake of clarity, we shall consider the $n \times 2n$ matrix A_k of 0, 1 and its corresponding sample space Ω_k. Therefore, in the sample space Ω_k we have that $Pr(U_1|L_1) = 1/4$ or $Pr(U_1|L_1) = 1/2$. Let us analyze a little bit more the induction hypothesis. The equation $(l_u, l_z) = (0, 1)$ holds for exactly $n/2$ points (columns) of A_k, since $Pr(l_u = 0 \wedge l_z = 1) = 1/4$ (due to pairwise independence). This means that there are either $n/8$ points of A_k such that $(l_v, l_w) = (0, 1)$ and $n/8$ points such that $(l_v, l_w) = (1, 0)$ (and thus $n/8$ points such that $(l_v, l_w) = (0, 0)$ and $n/8$ points such that $(l_v, l_w) = (1, 1)$), or $n/4$ points of A_k such that $(l_v, l_w) = (0, 1)$ and $n/4$ points such that $(l_v, l_w) = (1, 0)$.

We shall prove the assertion of the lemma for $n = 2^{k+1}$, i.e. for the sample space Ω_{k+1}. (If n is not a power of 2, we can just delete the last $2^{k+1} - n$ rows of A_{k+1}.) We distinguish among the following cases.

<u>Case 1:</u> Fix two vertices u, z such that $u, z \in [1, 2^k]$.

(1.a) $v, w \in [1, 2^k]$. Because of the pairwise independence, we have $(l_u, l_z) = (0, 1)$ in $n/2$ columns (points) of A_{k+1}, where half of them belong to A_k^I and the other half to A_k^{II}. By the induction hypothesis we have $(l_v, l_w) = (0, 1)$ either for $n/16$ or $n/8$ columns for each of A_k^I, A_k^{II}. Thus, for the new space Ω_{k+1} we have $\frac{n/16+n/16}{n/2} \leq Pr(U_1|L_1) \leq \frac{n/8+n/8}{n/2}$, i.e. the assertion of the lemma holds.

(1.b) $v, w \in [2^k + 1, 2^{k+1}]$

(1.b.1) $v = 2^k + u, w = 2^k + z$. Then (by the pairwise independence) we have $(l_u, l_z) = (0, 1)$ in $n/2$ columns of A_{k+1} and $(l_v, l_w) = (0, 1)$ in $n/4$ columns (of A_{k+1})which belong solely to A_k^{III}. We cannot have $(l_v, l_w) = (0, 1)$ in $\overline{A_k}$ (while $(l_u, l_z) = (0, 1)$) because of the following fact: suppose that $(l_v, l_w) = (0, 1)$ in column j, $1 \leq j \leq 2^{k+1}$ (i.e. in A_k^{III}). Then in the column $2^{k+1} + j$ (in which $(l_u, l_z) = (0, 1)$) we shall have by the construction of $\overline{A_k}$ $(l_v, l_w) = (1, 0)$. Thus, $Pr(U_1|L_1) = 1/2$ and the assertion holds.

(1.b.2) $v \neq 2^k + u, w \neq 2^k + z$. There are $n/2$ columns with $(l_u, l_z) = (0, 1)$, where half of them belong to A_k^I and half of them to A_k^{II}. This is the same if we were restricted: 1) into A_k^I where $v_{new} = v - 2^k$, $w_{new} = w - 2^k$ and looking for $(l_{v_{new}}, l_{w_{new}}) = (0, 1)$ while $(l_u, l_z) = (0, 1)$, and 2) into A_k^{III} where $v_{new} = v - 2^k$, $w_{new} = w - 2^k$ and looking for $(l_{v_{new}}, l_{w_{new}}) = (1, 0)$ while $(l_u, l_z) = (0, 1)$. (This last one happens because of $\overline{A_k}$). The rest of the proof now, is similar to case 1.a.

(1.b.3) $v \neq 2^k + u$, $w = 2^k + z$. Consider the first half of w. Then we are in the following

case: $u, z, v, w \in A_k^I$, $u \neq v \neq z$, $w = z$. Then, for the row $w = z$ there are $n/4$ 1's. From the pairwise independence, these $n/4$ 1's correspond to $n/8$ 0's and $n/8$ 1's of row v. This means that there are $n/8$ $(l_v, l_w) = (0, 1)$. For the other half of w, there are no pairs $(l_v, l_w) = (0, 1)$. The reason is the construction of $\overline{A_k}$, because in every column where $(l_u, l_z) = (0, 1)$ (i.e. $l_z = 1$) we have $l_w = 0$. Thus, $Pr(U_1|L_1) = \frac{n/8}{n/2} = 1/4$ and the lemma holds.

(1.b.4) $v = 2^k + u$, $w \neq 2^k + z$. Similar to 1.b.3.

(1.c) $v \in [1, 2^k]$, $v \neq u \neq z$, $w \in [2^k + 1, 2^{k+1}]$. Consider the row w. The first half of w, denoted as w_1, belongs to A_k^{III}. This means that we can restricted into A_k^I. Thus, either $n/16$ or $n/8$ columns have $(l_v, l_w) = (0, 1)$ by the induction hypothesis. The second half of w, denoted as w_2, belongs to $\overline{A_k}$. Therefore, we can be restricted into A_k^{II}, but now looking for $(l_v, l_w) = (0, 0)$. This can be done by taking as w_2 the complement of the real w_2. By the induction hypothesis there are either $n/16$ columns with $(l_v, l_w) = (0, 0)$ (corresponding to the case where there are $n/16$ columns having $(l_v, l_w) = (0, 1)$ in A_k^I), or none column with $(l_v, l_w) = (0, 0)$ (corresponding to the case where there are $n/8$ columns having $(l_v, l_w) = (0, 1)$ in A_k^I). Thus, the lemma holds in this case too.

Case 2: Fix two vertices u, z such that $u, z \in [2^k + 1, 2^{k+1}]$. The subcases of case 2 and their proofs are similar to those of case 1.

Case 3: Fix two vertices u, z, such that: either $u \in [1, 2^k]$ and $z \in [2^k + 1, 2^{k+1}]$ or $u \in [2^k + 1, 2^{k+1}]$ and $z \in [1, 2^k]$. Again the subcases of case 3 and their proofs are similar to the above ones. ∎

We are now ready to prove our main result.

Theorem 1 *Let σ_1^2, σ_2^2 be the variances of the r.v. $\widetilde{W}$ in the spaces (Ω_1, Pr_1) and (Ω_2, Pr_2) respectively. Then, $\sigma_2^2 \geq \sigma_1^2$.*

Proof: Let $e = \{v, w\}$. The variance σ^2 (in either space) is given by

$$\sigma^2 = E[(\sum_{e \in E} COST(l_v, l_w))^2] - (\frac{1}{2} \sum_{e \in E} w(e))^2 \quad (3)$$

Now, since $(\sum_{e \in E} COST(l_v, l_w))^2 = \sum_{e \in E} COST^2(l_v, l_w) + 2 \sum COST(l_v, l_w) \cdot COST(l_u, l_z)$ (where the second sum runs over all $\{v, w\}, \{u, z\} \in E$ such that $\{v, w\} \neq \{u, z\}$) and since $E[\sum_{e \in E} COST^2(l_v, l_w)] = \frac{1}{2} \sum_{e \in E} w^2(e)$ (in both spaces), we get finally (by (3)) the following result, where $e = \{v, w\}, e' = \{u, z\}$ and $A = \{\{e, e'\} : e \neq e' \wedge e, e' \in E\}$:

$$\sigma^2 = \frac{1}{4} \sum_{e \in E} w^2(e) - \frac{1}{2} \sum_{\{e, e'\} \in A} w(e) \cdot w(e') + 2 \sum_{\{e, e'\} \in A} E[COST(l_v, l_w) \cdot COST(l_u, l_z)] \quad (4)$$

Let $Q = \sum E[COST(l_v, l_w) \cdot COST(l_u, l_z)]$ where $\{v, w\} \in E, \{u, z\} \in E$ and $\{v, w\} \neq \{u, z\}$. Let Q_1, Q_2 be the values of Q in (Ω_1, Pr_1) and (Ω_2, Pr_2). We now calculate Q in the two spaces.

Space (Ω_1, Pr_1): To calculate Q_1, we split it into two parts, $Q_1 = P_{11} + P_{12}$ where

$$P_{11} = \sum E[COST(l_v, l_w) \cdot COST(l_u, l_z)]$$

running over all $\{v, w\}$ and $\{u, z\}$ with $v = u$, and P_{12} is a similar sum running over all $\{v, w\}$ and $\{u, z\}$ with distinct v, w, u, z. Clearly, in the case of P_{11},
$E[COST(l_v, l_w) \cdot COST(l_u, l_z)] =$
$= w(e) \cdot w(e') \cdot [Pr(l_v = 0 \wedge l_w = 1 \wedge l_z = 1) + Pr(l_v = 1 \wedge l_w = 0 \wedge l_z = 0)] = \frac{1}{4} w(e) \cdot w(e')$ (because of independence of l_v, l_w, l_z). Thus, $P_{11} = \frac{1}{4} \sum w(e) \cdot w(e')$ over all e, e' with one vertex in common. Similarly, $P_{12} = \frac{1}{4} \sum w(e) \cdot w(e')$ over all e, e' with disjoint vertices. Thus, the variance in the space (Ω_1, Pr_1) can be computed straightforwardly with only 4-wise independence.

Space (Ω_2, Pr_2): Again, we split Q_2 into two sums, $Q_2 = P_{21} + P_{22}$ where P_{21} runs over all pairs of edges with one vertex in common and P_{22} runs over all pairs of edges with disjoint vertices. Due to 3-wise independence in (Ω_2, Pr_2), it is not difficult to see that $P_{21} = P_{11}$.

For P_{22} we have: $P_{22} = \sum E[E[COST(l_v, l_w) | COST(l_u, l_z)] \cdot COST(l_u, l_z)]$ (5)

where the sum runs over all $\{v, w\}, \{u, z\}$ with distinct v, w, u, z.

But, $E[COST(l_v, l_w) | COST(l_u, l_z)]$ can be computed using the the conditional probabilities $Pr(U_1|L_1), Pr(U_1|L_2), Pr(U_2|L_1), Pr(U_2|L_2)$, which in turn are provided by lemma 1. Therefore, (by lemma 1 and formula (5)) we have that $\frac{1}{4} \sum w(e) \cdot w(e') \leq P_{22} \leq \frac{1}{2} \sum w(e) \cdot w(e')$, where $e, e' \in E$ and have no points in common. Hence, $P_{12} \leq P_{22}$ which implies that $Q_1 \leq Q_2$. Now, by formula (4), we get $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_1^2 + \frac{1}{4} \sum w(e) \cdot w(e')$, which completes the proof. ∎

We prove now that we can apply Chebyshev's inequality on the r.v. $\widetilde{W}$ of (Ω_1, Pr_1) using the variance in the space (Ω_2, Pr_2).

Theorem 2 *For the r.v.* $\widetilde{W}$ *of* (Ω_1, Pr_1) *we have* $Pr(\ |\widetilde{W} - \overline{W_2}| \geq t \cdot \sigma_2\ ) \leq \frac{1}{t^2}$, *for any* $t > 0$.

Proof: Since $\overline{W_1} = \overline{W_2}$ and $\sigma_1^2 \leq \sigma_2^2$, we have $|\widetilde{W} - \overline{W_2}| \geq t \cdot \sigma_2 \Rightarrow |\widetilde{W} - \overline{W_1}| \geq t \cdot \sigma_2 \geq t \cdot \sigma_1$. Let E_1, E_2 be events defined as follows: $E_1 =$ "$|\widetilde{W} - \overline{W_1}| \geq t \cdot \sigma_1$" and $E_2 =$ "$|\widetilde{W} - \overline{W_2}| \geq t \cdot \sigma_2$". Then $E_2 \Rightarrow E_1$, thus $Pr(E_1|E_2) = 1 \Rightarrow Pr(E_1) \geq Pr(E_2)$. But $Pr(E_1) \leq \frac{1}{t^2}$ by Chebyshev's inequality. Thus $Pr(E_2) \leq \frac{1}{t^2}$ also. ∎

4 The NC Approximation Algorithm

The following algorithm constructs in NC the small space Ω_2 and computes the mean and the variance of $\widetilde{W}$ in Ω_2.

(1) **for all** $\omega \in \Omega_2$ and $v \in V$ *compute in parallel* $l_v(\omega)$;

(2) **for all** $\omega \in \Omega_2$ *compute in parallel* $W(C(\hat{l}(\omega)))$;

(3) *compute* $\overline{W_2}$; (* using formula (2) *)

(4) *compute* σ_2^2 ; (* using formula (3) *)

Theorem 3 *The above algorithm constructs the $\Theta(n)$-sized space Ω_2 and computes the mean and the variance of $\widetilde{W}$ in Ω_2, in $O(\log n)$ time using $O(n^2 + nm/\log n)$ processors on an EREW PRAM and thus provides a (tight) upper bound on the fraction of edge cuts whose weight is greater than a given integer.*

Proof: The correctness of the algorithm follows by previous discussion. Clearly, all the steps of the above algorithm can be computed in $O(\log n)$ time, with at most $O(n^2 + nm/\log n)$ processors, using an optimal parallel prefix sum algorithm (see, e.g. [3]). ■

5 Conclusions and Further Work

The fact that $\sigma_2^2 \neq 0$ and also that the mean value of the r.v. $\widetilde{W}$ in Ω_2 is equal to $\frac{1}{2}\sum_{e\in E} w(e)$, implies that there is an edge cut in Ω_2 with weight *strictly bigger* than the mean. Since the maximum cut is at most $\sum_{e\in E} w(e)$, we have a $(1/2+\varepsilon)$ (for some $\varepsilon > 0$) approximation solution to the max cut problem. Clearly, this solution can be obtained within the resource bounds of theorem 3.

We do not have an analogous main lemma (as lemma 1) for higher moments, because the corresponding conditional probability becomes zero for a *small* number of vertices. Thus, the existence of a result like lemma 1 depends now on the weights of the edges of the graph. In the special case where the values of all the weights are almost the same, we can get a similar result to theorem 1 between the third moments in the two spaces. We conjecture that, when the largest difference between the edge weights is $o(n)$, then theorem 1 holds for all the higher moments. In such a case, tighter bounds would exist (and the smaller space would approximately *characterize the distribution* of the cut weights of the graph). It seems that our approximation scheme can be extended to handle other NP-hard and/or #P-complete problems (especially graph partitions and weighted path problems).

Acknowledgements. We are grateful to Spyros Tragoudas for pointing out the applications of the max cut problem.

References

[1] N. Alon, L. Babai and A. Itai, "A fast and simple Randomized parallel Algorithm for the Maximal Independent Set Problem", J. of Algorithms, 7, pp. 567-583, 1986.

[2] B. Berger, "Using randomness to design Efficient Deterministic Algorithms", Ph.D. Thesis, Dept. of Electrical Engineering and Computer Science, MIT 1990.

[3] R. Cole and U. Vishkin, "Deterministic Coin Tossing with Applications to Optimal Parallel List Ranking", Inform. and Control, Vol.70, No.1, pp. 32-53, 1986.

[4] M. Garey and D. Johnson, "Computers and Intractability – A Guide to the Theory of NP-Completeness", Ed. Freeman and Co, San Francisco, 1979.

[5] M. Garey, D. Johnson and L. Stockmeyer, "Some Simplified NP-Complete Graph Problems", Theor. Comp. Science 1, pp. 237-267, 1976.

[6] M. Goldberg and T. Spencer, "A new Parallel Algorithm for the Maximal Independent Set Problem", SIAM J. on Computing, pp. 419-427, 1989.

[7] R. Karp and A. Wigderson, "A Fast Parallel Algorithm for the Maximal Independent Set Problem", Proc. 16th ACM STOC, pp. 266-272, 1984.

[8] C. H. Lee, M. Kim and C. I. Park, "An efficient k-way partitioning algorithm for task allocation in parallel computing systems", Proc. of Int. Conf. on Systems Integration, pp. 748-751, 1990.

[9] M. Luby, "A Simple Parallel Algorithm for the Maximal Independent Set Problem", Proc. 17th ACM STOC, pp. 1-10, 1985.

[10] M. Luby, "Removing Randomness in Parallel Computation without a Processor Penalty", Proc. 29th IEEE FOCS, pp. 162-174, 1988.

[11] G. Pantziou, P. Spirakis and C. Zaroliagis, "Fast Parallel Approximations of the Maximum Weighted Cut Problem through Derandomization", Computer Technology Institute, Technical Report, TR-89.04.03, April 1989.

[12] Naveed Sherwani, "Algorithms for VLSI Physical Design Automation", Kluwer Academic Publishers, 1993.

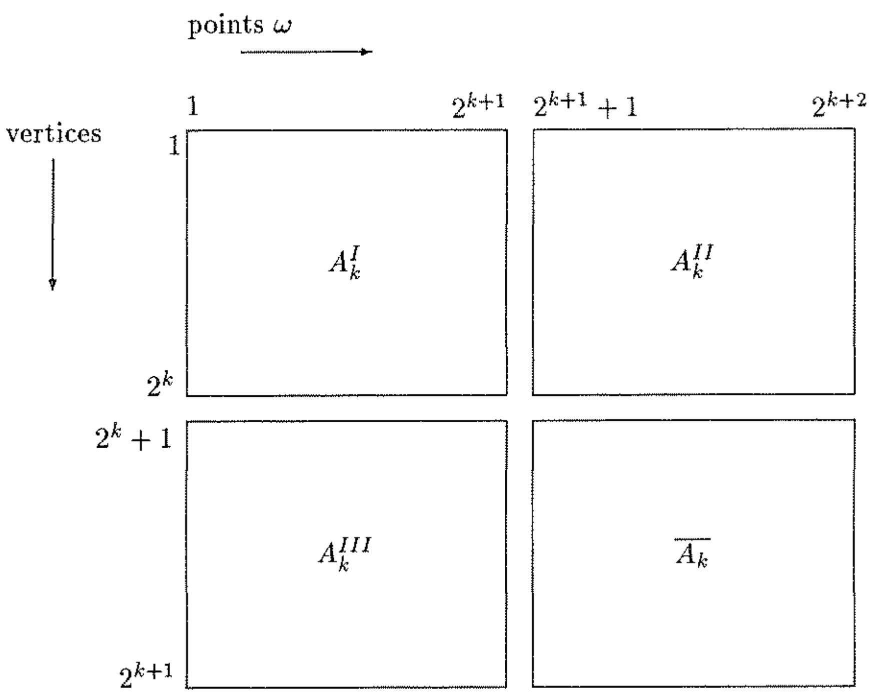

Figure 1: Recursive construction of the smaller sample space.