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Spectral Sequences and Khovanov Homology

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SPECTRAL SEQUENCES AND KHOVANOV HOMOLOGY

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Abstract

In this thesis, we will focus on two main topics; the common thread between both will be the existence of spectral sequences relating Khovanov homology to other knot invariants.

Our first topic is an invariant $\mathrm{MKh}(L)$ for links in thickened disks with multiple punctures. This invariant is different from but inspired by both the Asaeda-Pryzytycki-Sikora (APS) homology and its specialization to links in the solid torus. Our theory will be constructed from a $\mathbb{Z}^n$-filtration on the Khovanov complex, and as a result we will get various spectral sequences relating $\mathrm{MKh}(L)$ to $\mathrm{Kh}(L)$, $\mathrm{AKh}(L)$, and $\mathrm{APS}(L)$.

Our second topic is the Dowlin spectral sequence, which has E_2-page isomorphic to the reduced Khovanov homology $\overline{\mathrm{Kh}}(L)$, and which converges to the knot Floer homology $\widehat{\mathrm{HFK}}(L)$ on the E_∞-page. While it was previously known that these two pages are link invariants, we prove that every page is an invariant by defining weak maps on the underlying filtered complex which correspond to Reidemeister moves. This result is based on joint work with Samuel Tripp.

Preface

I would like to thank my advisor, Ina Petkova, for her guidance and encouragement. Ina's enthusiasm for low-dimensional topology is infectious, and kindled my own love of knot theory. She patiently worked with me to find projects that interested both of us, and I credit her with showing me the beauty of my eventual thesis topic.

Additionally, I would like to thank John Baldwin, Vladimir Chernov, and Michael C.-M. Wong for serving on my thesis committee. I would also like to thank them, along with Nate Dowlin, Eli Grigsby, and Mikhail Khovanov, for their advice and thought-provoking conversation over the years.

I would also like to thank the other members of my grad school cohort: Matt Jones, Sam Tripp, and Yao Xiao. Thanks to Sam, in particular, for working with me on the material in the second half of this thesis. More generally, I would like to thank the entire mathematics community at Dartmouth for their support these past five years.

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Finally, I would like to thank my wife, Kelsey Winkeler, who pushed me to apply for grad school when I was uncertain of myself, and spent countless hours over the years proofreading my various writings (including this preface). Only with her love and support, as well as the rest of my family and friends, was I able to reach this point.

Contents

Chapter 1

Introduction

Originally defined in [22], Khovanov homology $\mathrm{Kh}(L)$ is an invariant of links $L \subset S^3$ categorifying the Jones polynomial. Khovanov homology is quite powerful, and has been applied to answer many different questions in knot theory. For example, in [49] Rasmussen used it to define the integer-valued s-invariant, which gives a combinatorial proof of the Milnor conjecture, and in general a lower bound on the slice genus of a knot K. Other applications of Khovanov homology include detection of quasi-alternating links [31] and an upper bound on the Thurston-Bennequin number of Legendrian knots [36]. Additionally, variations on the construction of $\mathrm{Kh}(L)$ have given rise to a whole subfield of knot invariants, such as Lee homology [28], $\mathfrak{sl}(n)$ homology [23, 21], and HOMFLY-PT homology [25], among many others.

In this thesis, we will focus on two main topics; the common thread between both will be the existence of spectral sequences relating Khovanov homology to other knot invariants. Spectral sequences are the form that many relationships between "categorified" knot invariants take. For example, spectral sequences exist from the HOMFLY-PT homology of a link to its $\mathfrak{sl}(n)$ homology [48], corresponding to the substitution $a = q^n$ turning the two-variable HOMFLY-PT polynomial into the one-variable $\mathfrak{sl}(n)$ polynomial. Spectral sequences also give rise to rank inequalities (e.g.

[56]), and the behavior of elements under them can be used to define new invariants (e.g. Rasmussen's aforementioned s-invariant, as well as its generalization to $\mathfrak{sl}(n)$ homology [30, 53]).

Our first topic involves extensions of Khovanov homology to links in manifolds other than S^3. One of the most well-known such generalizations is the annular Khovanov homology $\mathrm{AKh}(L)$. This theory takes as input a link $L \subset A \times I$ in the thickened annulus, and uses this extra embedding information to construct a filtration on the Khovanov complex $\mathrm{CKh}(L)$. The invariant $\mathrm{AKh}(L)$ is then computed as the homology of the associated graded complex [50]. Aside from just being an invariant of annular links, $\mathrm{AKh}(L)$ has given rise to invariants of braid conjugacy classes [16, 19], and has been shown to distinguish braids from other tangles [17, 54]. There is a natural $\mathfrak{sl}_2$ action [15] on $\mathrm{AKh}(L)$, which generalizes to an $\mathfrak{sl}_n$ action on Khovanov-Rozansky homology [44]. One can define a Lee-type deformation of annular Khovanov homology as well [16].

The invariance of annular Khovanov homology under Reidemeister moves was actually first proved in a different context: Asaeda, Przytycki, and Sikora described a homology theory $\mathrm{APS}(L)$ for framed links in thickened surfaces $\Sigma \times I$ which involved a grading on $\mathrm{CKh}(L)$ [2]. This grading essentially records the link diagram resolutions up to homotopy. It turns out that when our surface is the annulus $\Sigma = A$, this grading filters the Khovanov complex. As a result, for annular links $\mathrm{APS}(L)$ is the same as $\mathrm{AKh}(L)$ (up to a grading shift to forget the framing).

This ability to describe $\mathrm{AKh}(L)$ in terms of filtrations is particularly powerful. For example, it gives us access to classical results about spectral sequences, so we automatically know that there is a spectral sequence from $\mathrm{AKh}(L)$ to $\mathrm{Kh}(L)$. If we view the annulus as a disk with a single puncture, we could ask if a similar construction exists for disks with multiple punctures. In Chapter 3, we define an invariant for links L in thickened disks with multiple punctures $\Sigma \times I$ analogous to $\mathrm{AKh}(L)$. While $\mathrm{APS}(L)$ already provides one invariant of such knots, the grading is by homotopy classes of loops; the invariant in this thesis instead takes the form of the filtered chain homotopy type of a complex, where the filtration is by $H_1(\Sigma;\mathbb{Z})$. Since our theory will be constructed from a $\mathbb{Z}^n$-filtration on the Khovanov complex, as a result we will get various spectral sequences relating $\mathrm{MKh}(L)$ to $\mathrm{Kh}(L)$, $\mathrm{AKh}(L)$, and $\mathrm{APS}(L)$.

Our second topic is joint work with Samuel Tripp, and it concerns the relationship between Khovanov homology and knot Floer homology. The latter is an invariant of links which has its origins in Heegaard Floer homology $\widehat{\mathrm{HF}}(M)$, an invariant of closed 3-manifolds first defined by Oszváth and Szabó in [41]. Later, the same authors [40] and Rasmussen [49] would independently discover that a knot $K \subset M$ in a 3-manifold M can be used to filter the Heegaard Floer chain complex $\widehat{\mathrm{CF}}(M)$, and that the associated graded complex with respect to this filtration is actually a knot invariant, known as knot Floer homology $\widehat{\mathrm{HFK}}(K)$. Unless specified otherwise, we will assume $M = S^3$ since we are interested in comparing $\widehat{\mathrm{HFK}}(K)$ and $\overline{\mathrm{Kh}}(K)$. Like Khovanov homology, knot Floer homology categorifies a knot polynomial: the Alexander polynomial. Also like Khovanov homology, knot Floer homology has produced many interesting and powerful results in knot theory; for example, it detects the genus [39] and fiberedness [38] of knots. The invariant $\widehat{\mathrm{HFK}}(K)$ also gives rise to a concordance invariant $\tau(K)$ [37, 46], which in many ways resembles the s-invariant from Khovanov homology. Due to these similarities and others, Rasmussen conjectured in [47] that there could be a spectral sequence from $\overline{\mathrm{Kh}}(K)$ to $\widehat{\mathrm{HFK}}(K)$. The existence of such an object was proven by Dowlin in [14], who constructed a spectral sequence with E_2-page isomorphic to the reduced Khovanov homology $\overline{\mathrm{Kh}}(L)$ that converges to the knot Floer homology $\widehat{\mathrm{HFK}}(L)$ on the E_∞-page.

The Dowlin spectral sequence has already been used to produce new results in Khovanov homology; for example, we now know that $\mathrm{Kh}(L)$ detects several more knots, like the figure-eight knot [7], the cinquefoil [6], and the $(2,6)$-torus knot [33]. It would be interesting to know if the spectral sequence relates pairs of invariants defined separately in Khovanov homology and knot Floer homology, such as the aforementioned s and τ.

Roberts was able to prove a similar result about the spectral sequence from $\mathrm{Kh}(L)$ to the Heegaard Floer homology of the double branched cover $\Sigma(L)$ of S^3 along L, specifically that it related the transverse knot invariant $\psi(L) \in \mathrm{Kh}(L)$ to the contact class $c(\xi_L) \in \widehat{\mathrm{HF}}(-\Sigma(L))$ [50]. Baldwin improved this result by showing the invariance of each page of the spectral sequence, and consequently constructing a whole family of invariants related to $\psi(L)$ [4].

While we know that the E_2- and E_∞-pages of the Dowlin spectral sequence are link invariants, since they are $\overline{\mathrm{Kh}}(L)$ and $\widehat{\mathrm{HFK}}(L)$ respectively, it is not immediate that every page in between should be. That the higher pages of the Dowlin spectral sequence are in fact link invariants is the main result of Chapter 4.

Organization. First, we review background information and establish conventions regarding links and their Khovanov homology that will be relevant to both of our main topics in Chapter 2. Next, we discuss $\mathrm{MKh}(L)$, our invariant of links in thickened disks with multiple punctures, in Chapter 3. Finally, we discuss our results about the invariance of the Dowlin spectral sequence in Chapter 4. At the end of the thesis we include Appendix A, which we will reference throughout for useful lemmas and proofs.

Chapter 2

Background

Section 2.1

Links

For the purposes of this paper, an n-component *link* is a smooth embedding of $\sqcup_n S^1$ into S^3, considered up to ambient isotopy. We will represent links via *diagrams*, which are 4-valent graphs with vertices labeled by over/under crossing data. We think of diagrams as projections of links onto $\mathbb{R}^2$ with no triple points or tangencies. All links are assumed to be *oriented*, which means that we have chosen one of the two possible orientations for each of the link components. Given an oriented link diagram, we can define a *positive crossing* as a part of a diagram that locally looks like , and a *negative crossing* as a part of a diagram that locally looks like .

Section 2.2

Khovanov homology

First defined in [22], Khovanov homology is an invariant of links in S^3. We will give a short description of its definition here; for a more detailed description, see the original paper or Bar-Natan's paper [11].

We will use two gradings to define this invariant. The first grading is the homological grading g^h; following Bar-Natan's convention, shifts in this grading will be denoted with brackets, i.e. $M[i]_n = M_{n-i}$. The second grading is the quantum grading g^q; shifts in this grading will be denoted with curly braces, i.e. $M\{i\}_n = M_{n-i}$.

Let L be a link in $\mathbb{R}^3 \subset S^3$. Choose a projection $\mathbb{R}^3 \to \mathbb{R}^2$ so that the image of L is a 4-valent graph D with crossing information at each vertex, i.e. a link diagram. Choose an ordering $c_1, \ldots, c_n$ of the crossings of D. Given a crossing , define its 0-resolution to be  and its 1-resolution to be) (. If $S \in \{0,1\}^n$ is any element, then define $D(S)$ to be the collection of closed circles in $\mathbb{R}^2$ obtained from D by replacing each crossing c_i with its S_i-resolution.

Fix a ring $\Bbbk$ (usually a field or $\mathbb{Z}$), and let $V = \Bbbk \langle v_+, v_- \rangle$ be the free $\Bbbk$-module generated by two elements v_+ and v_-. We define the quantum gradings of v_+ and v_- to be 1 and -1, respectively. To a fully-resolved diagram $D(S)$ with k components, we associate a module $\mathrm{CKh}(D(S)) = V^{\otimes k}$. We think of a generator $x \in V^{\otimes k}$ as representing a labeling of each circle $C_i \in D(S)$ with a sign $s_i \in \{+,-\}$. We call such a labeling $\{(C_i, s_i)\}_i$ an *enhanced (Kauffman) state*. Let n_+ and n_- be the number of positive crossings and negative crossings in D, respectively, and let $\mathrm{wr}(D) = n_+ - n_-$ be the *writhe* of D. To the original diagram D, we associate a module:

$$\mathrm{CKh}(D) := \bigoplus_{S \in \{0,1\}^n} \mathrm{CKh}(D(S))[|S| - n_-]\{|S| - n_- + \mathrm{wr}(D)\}$$

When we care about the gradings, we will write $\mathrm{CKh}^{h,q}(D)$ to denote the graded piece of $\mathrm{CKh}(D)$ with homological grading h and quantum grading q.

We can give $\mathrm{CKh}(D)$ the structure of a chain complex by defining a differential $\partial : \mathrm{CKh}(D) \to \mathrm{CKh}(D)$ on it. Let $S \in \{0,1\}^n$, and let $x \in \mathrm{CKh}(D(S))$ be a generator. Let $S' \in \{0,1\}^n$ be obtained from S by changing a single 0 to a 1. If $D(S)$ has k components, then $D(S')$ has either $k-1$ or $k+1$ components; we call the former a *merge* and the latter a *split*. If S and S' are related by a merge, then we can define a map $m : \mathrm{CKh}(D(S)) \cong V^{\otimes k} \to V^{\otimes k-1} \cong \mathrm{CKh}(D(S'))$. On the two tensor factors that are being merged, m is defined as

$$m : V \otimes V \to V = \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0 \end{cases}$$

We extend m to all of $\mathrm{CKh}(D(S))$ by setting it to be the identity on all the other tensor factors. Similarly, if S and S' are related by a split, we define a map Δ : $\mathrm{CKh}(D(S)) \cong V^{\otimes k} \to V^{\otimes k+1} \cong \mathrm{CKh}(D(S'))$. On the tensor factor that is being split, Δ is defined as

$$\Delta : V \to V \otimes V = \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

We extend Δ to all of $\mathrm{CKh}(D(S))$ be setting it to be the identity on all the other tensor factors. Let $\partial_{S,S'} : \mathrm{CKh}(D(S)) \to \mathrm{CKh}(D(S'))$ denote the map m if S and S' are related by a merge, Δ if S and S' are related by a split, and 0 otherwise. We will implicitly extend the domain and codomain of $\partial_{S,S'}$ by zero when we define the Khovanov differential $\partial : \mathrm{CKh}(D) \to \mathrm{CKh}(D)$ as

$$\partial := \sum_{S,S' \in \{0,1\}^n} \epsilon(S,S') \partial_{S,S'}$$

Here $\epsilon(S, S') \in \{1, -1\}$ is some sign-assignment function to ensure that $\partial^2 = 0$. There are many possible functions to choose for ϵ; one that works here is to let

$$\epsilon(S, S') := (-1)^{\sum_{i=1}^{j-1} S_i}$$

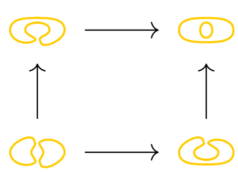
where j is the first index at which $S_i \neq S'_i$. The Khovanov differential has degree 1 with respect to the homological grading, and 0 with respect to the quantum grading.

Finally, we define the *Khovanov homology* of L (with coefficients in $\Bbbk$) as the homology of this chain complex[1]:

$$\mathrm{Kh}(L; \Bbbk) := H^*(\mathrm{CKh}(D), \partial)$$

We will often omit writing the coefficient ring if it is clear from context. One can prove that Kh is an invariant of the link L, and therefore does not depend on the choice of diagram D, ordering of the crossings $c_1, \ldots, c_n$, or sign assignment ϵ.

Example 2.2.1. Let $L =$  be the Hopf link. Orient both components counter-clockwise. Choose the base ring $\Bbbk = \mathbb{Q}$, so $V = \mathbb{Q}^2$. We use the cube of resolutions to form the chain complex $\mathrm{CKh}(L)$:



$$\begin{array}{ccc} V & \xrightarrow{\Delta_2} & V^{\otimes 2} \\ {\scriptstyle m_1}\uparrow & & \uparrow{\scriptstyle \Delta_1} \\ V^{\otimes 2} & \xrightarrow{m_2} & V \end{array}$$

We can write the above in a "flatter" version:

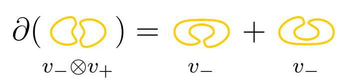
$$V^{\otimes 2} \xrightarrow{\binom{m_1}{m_2}} V \oplus V \xrightarrow{(-\Delta_2\ \Delta_1)} V^{\otimes 2}$$

Replacing V, m, and Δ by their definitions gives:

[1] Since the differential has degree 1, it would perhaps be more appropriate to refer to $\mathrm{CKh}(D)$ as a *cochain* complex, and thus we would be taking its *cohomology*. This justifies our use of superscripts, but we will continue to use words like "chain complex" and "homology", as this seems to be the convention.

$$\mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 1&0&0&0\\ 0&1&1&0\\ 1&0&0&0\\ 0&1&1&0 \end{pmatrix}} \mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 0&0&0&0\\ -1&0&1&0\\ -1&0&1&0\\ 0&-1&0&1 \end{pmatrix}} \mathbb{Q}^4$$

Note that we had to choose an ordered basis of each $V^{\otimes n}$ to write the matrices above. The convention we will use in this paper is that tensor factors are sorted left-to-right by the leftmost points in the loops representing them. Additionally, the signs labelling each loop start at all + signs and count up in binary (e.g. $++,+-,-+,--$). For example, the 3rd column of the left matrix above says that



Since our diagram contains no positive crossings and two negative crossings, we see that $n_+ = 0$, $n_- = -2$, and $\mathrm{wr}(D) = -2$, which tells us how to shift the gradings. To find the Khovanov homology of the Hopf link, we take the homology of the Khovanov complex to find that:

$$\begin{aligned} \mathrm{Kh}^{-2}(L) &\cong \mathbb{Q}\{-4\} \oplus \mathbb{Q}\{-6\} \\ \mathrm{Kh}^{-1}(L) &\cong 0 \\ \mathrm{Kh}^{0}(L) &\cong \mathbb{Q} \oplus \mathbb{Q}\{-2\} \end{aligned}$$

Chapter 3

Multipunctured Khovanov homology

Section 3.1

Introduction

In this chapter, we use a generalized notion of filtration over an arbitrary poset. This construction has been studied before, for example in [34] and [18]. We use these generalized filtrations to define a homology theory $\mathrm{MKh}(L)$ for links in thickened disks with multiple punctures. Specifically, given a link diagram D for a link L in a thickened n-punctured disk $\Sigma \times \mathbb{I}$, we construct a $\mathbb{Z}^n$-filtration on the Khovanov complex $\mathrm{CKh}(D)$. This filtration essentially records the annular grading of a generator with respect to each of the n punctures. We then define $\mathrm{MKh}(L)$ to be the homology of the associated graded complex. This homology is naturally graded by $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^n$, as it inherits the homological and quantum gradings from Khovanov homology, as well as the multipunctured grading used to construct the filtration. Given a homological grading $i \in \mathbb{Z}$, a quantum grading $j \in \mathbb{Z}$, and a multipunctured grading $v \in \mathbb{Z}^n$, we denote the corresponding graded piece $\mathrm{MKh}^{i,j,v}(L)$. Therefore, the whole homology decomposes as:

$$\mathrm{MKh}(L) = \bigoplus_{i,j,v} \mathrm{MKh}^{i,j,v}(L)$$

In Section 3.7, we prove that this is in fact an invariant:

Theorem 3.1.1. *Let $L \subset \Sigma \times \mathbb{I}$ be a link in a thickened n-punctured disk, and let $\Bbbk$ be a commutative ring. Then $\mathrm{MKh}(L;\Bbbk)$ is an invariant of such links, as a triply-graded $\Bbbk$-module. In fact, for any diagram D of L, we get that the filtered chain homotopy type of $\mathrm{CKh}(D;\Bbbk)$ is an invariant of L.*

While APS homology is already defined for band links in thickened surfaces, our theory differs in that it is defined for unframed links, and is graded by $H_1(\Sigma;\mathbb{Z})$, the first integral homology group of the surface instead of by $\mathbb{Z}C(\Sigma)$, the free abelian group on homotopy classes of simple closed curves. Additionally, since $\mathrm{MKh}(L)$ is computed from a filtered chain complex, we have access to more filtration-related structure.

Despite their differences, we still have a relationship between $\mathrm{APS}(L)$ and $\mathrm{MKh}(L)$. In order to compare the two, we define a homology $\widetilde{\mathrm{APS}}(L)$ for links in thickened n-punctured disks, which is related to $\mathrm{APS}(L)$ by a change in gradings, and is an invariant of (unframed) links. This invariant is also a triply-graded module; here, $i \in \mathbb{Z}$ is the homological grading, $j \in \mathbb{Z}$ is the quantum grading, and $v \in \mathbb{Z}C(\Sigma)$ is the negative of the Ψ-grading from [2] (analogous to our multipunctured grading).

$$\widetilde{\mathrm{APS}}(L) = \bigoplus \widetilde{\mathrm{APS}}^{i,j,v}(L)$$

Define the graded Euler characteristics of $\widetilde{\mathrm{APS}}(L)$ and $\mathrm{MKh}(L)$ as

$$\chi(\widetilde{\mathrm{APS}}(L)) := \sum_{i,j,v} (-1)^i q^j x^v \operatorname{rk}(\widetilde{\mathrm{APS}}^{i,j,v}(L))$$
$$\chi(\mathrm{MKh}(L)) := \sum_{i,j,v} (-1)^i q^j y^v \operatorname{rk}(H^{i,j,v}(L))$$

Here, x^v and y^v are shorthand notations for certain products of variables x_c and y_c, one variable for each basis element of $\mathbb{Z}C(\Sigma)$ and $H_1(\Sigma;\mathbb{Z})$, respectively. See Section 3.6.2 for details.

We then get that these two Laurent polynomials are related by a Hurewicz-like map $h : \mathbb{Z}C(\Sigma) \to H_1(\Sigma;\mathbb{Z})$.

Theorem 3.1.2. *The map h induces a ring map $\chi_h : \mathbb{Z}[q^{\pm 1}, x^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}, y^{\pm 1}]$, which sends $\chi(\widetilde{\mathrm{APS}}(L))$ to $\chi(MKh(L))$.*

We then construct a spectral sequence from $\widetilde{\mathrm{APS}}(L)$ to $\mathrm{MKh}(L)$. While our previous constructions use filtrations and gradings over $\mathbb{Z}^n$, we first need to "flatten" these to $\mathbb{Z}$-filtrations and $\mathbb{Z}$-gradings to make use of the conventional machinery of spectral sequences associated to filtered chain complexes.

Theorem 3.1.3. *Let Σ be a disk with n punctures, and let L be a link in $\Sigma \times \mathbb{I}$. Then there is a spectral sequence with E_1 page isomorphic to $\widetilde{\mathrm{APS}}(L)$ that converges to $\mathrm{MKh}(L)$ (with flattened gradings).*

On the other hand, our construction is in some ways like a generalization of annular Khovanov homology[1]. We identify $\mathrm{AKh}(L)$ as a special case of $\mathrm{MKh}(L)$, and construct a spectral sequence from the latter to the former:

Theorem 3.1.4. *Let Σ be a disk with $n > 1$ punctures, and let L be a link in $\Sigma \times \mathbb{I}$. Choose a puncture p, and let $A = \mathbb{D}^2 \setminus \{p\}$ be the annulus obtained from Σ by filling*

[1]Which therefore makes our theory a generalization of a specialization of a generalization of Khovanov homology.

in every puncture except p. Then there is a spectral sequence with E_1 page isomorphic to $\mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$ *(with flattened grading) that converges to* $\mathrm{AKh}(L \subset A \times \mathbb{I})$.

We also recover the well-known spectral sequence from $\mathrm{AKh}(L)$ to $\mathrm{Kh}(L)$, as well as a spectral sequence from $\mathrm{MKh}(L)$ to $\mathrm{Kh}(L)$:

Theorem 3.1.5. *For any link L in an n-punctured disk Σ, there is a spectral sequence with E_1 page isomorphic to* $\mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$ *(with flattened grading) that converges to* $\mathrm{Kh}(L)$.

Theorem 3.1.4 and Theorem 3.1.5 are special cases of a more general relationship between links that are related by thickened surface embeddings.

Theorem 3.1.6. *Let Σ be a disk with n punctures, and let Σ' be a disk with a subset of those n punctures. Let $L \subset \Sigma \subset \Sigma'$ be a link. Then there is a spectral sequence with E_1 page isomorphic to* $\mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$ *that converges to* $\mathrm{MKh}(L \subset \Sigma' \times \mathbb{I})$ *(with flattened gradings).*

In [45], Queffelec and Wedrich define a functor from links in thickened surfaces to a category of foams which allows one to recover $\mathrm{Kh}(L)$, $\mathrm{AKh}(L)$, and $\mathrm{APS}(L)$, among other link invariants. They also exhibit similar spectral sequences corresponding to surface embeddings. It seems likely that $\mathrm{MKh}(L)$ has some relationship with this construction.

In another direction, $\mathrm{MKh}(L)$ may be useful for extracting information about braids from $\mathrm{AKh}(L)$. Stabilization and destabilization of braids can't be realized by ambient isotopy in $A \times \mathbb{I}$, but they can be viewed as a sequence of adding and removing punctures to a diagram. These moves would correspond to spectral sequences between various values of $\mathrm{MKh}(L)$.

Since we started out trying to generalize annular Khovanov homology, it's natural to ask if $\mathrm{MKh}(L)$ is any better at distinguishing knots in thickened multipunctured disks. It turns out that it is, i.e. we can find knots $L, L' \subset \Sigma \times \mathbb{I}$ that have isomorphic $\mathrm{AKh}(L \subset A \times \mathbb{I})$ around each puncture individually, but different $\mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$. We give one such example in Section 3.8.

Organization

In Section 3.2, we give some background information on annular Khovanov homology. In Section 3.3, we discuss some facts about filtrations by posets. We define our main invariant $\mathrm{MKh}(L)$ in Section 3.4, then discuss the natural spectral sequences it comes equipped with in Section 3.5. In Section 3.6, we relate $\mathrm{MKh}(L)$ with Khovanov homology, annular Khovanov homology, and APS homology. In Section 3.7, we prove that $\mathrm{MKh}(L)$ is in fact a link invariant. Finally, we give an example to demonstrate how $\mathrm{MKh}(L)$ can be non-trivial even when $\mathrm{AKh}(L)$ is trivial in Section 3.8.

Acknowledgements

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Section 3.2

Annular Khovanov homology

We will begin by giving a definition of annular Khovanov homology.

Let $A = \mathbb{D}^2 \setminus *$ be an annulus. If D is a diagram for a link $L \subset A \times I$, then one can define more structure on the Khovanov complex $(\mathrm{CKh}(D), \partial)$. If $x = \{(C_i, s_i)\}_i$ is an enhanced state of $\mathrm{CKh}(D)$, let

$$k(x) := \sum_{i=1}^{j} s_i \, [C_i] \qquad \in H_1(A; \mathbb{Z})$$

Since $H_1(A;\mathbb{Z}) \cong \mathbb{Z}$, k actually specifies a grading on the underlying module $\mathrm{CKh}(L)$. In [50], Roberts proved that the filtration $\mathcal{F}^A$ induced by k on $(\mathrm{CKh}(L), \partial)$ is compatible with the Khovanov differential, and thus we can view $(\mathrm{CKh}(L), \partial, \mathcal{F}^A)$ as a filtered chain complex. The *annular Khovanov homology* of L is then defined to be

$$\mathrm{AKh}(L) \coloneqq H^*(\mathrm{gr}(\mathrm{CKh}(L), \partial, \mathcal{F}^A))$$

That the above is actually an invariant of annular links was first proved by Asaeda, Przytycki, and Sikora in [2], although not in those exact words.

Example 3.2.1. Let $L =$ ; here, the black dot represents the missing center of the annulus. We construct a similar cube of resolutions as before, then flatten to obtain the same Khovanov chain complex as in Example 2.2.1, this time with a filtration. We consider the associated graded complex:

$$\mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 1&0&0&0\\0&1&1&0\\1&0&0&0\\0&1&1&0 \end{pmatrix}} \mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 0&0&0&0\\-1&0&1&0\\-1&0&1&0\\0&0&0&0 \end{pmatrix}} \mathbb{Q}^4$$

One can think of the differential on this complex as the original Khovanov differential, but without the two entries

since these lower the k-grading. This complex allows us to calculate that:

$$\mathrm{AKh}^{-2}(L) \cong \mathbb{Q}\{-4\} \oplus \mathbb{Q}\{-6\}$$
$$\mathrm{AKh}^{-1}(L) \cong \mathbb{Q}\{-4\}$$
$$\mathrm{AKh}^{0}(L) \cong \mathbb{Q} \oplus \mathbb{Q}\{-2\} \oplus \mathbb{Q}\{-4\}$$

Section 3.3

Filtrations by Posets

In this section, we generalize the notion of a filtration from Appendix A.1; instead of indexing subobjects over $\mathbb{Z}$, we allow filtrations by arbitrary posets $\mathcal{P}$ subject to some conditions. This general notion of filtration will be used to define our main invariant in Section 3.4.

3.3.1. $\mathcal{P}$-filtrations

Recall that a *partially-ordered set* (usually abbreviated "poset") is a set $\mathcal{P}$ equipped with a partial order $\leq$, i.e. a relation that is reflexive, anti-symmetric, and transitive. A poset is *bounded* if there exists a least element $\perp \in \mathcal{P}$ and a greatest element $\top \in \mathcal{P}$. A map of posets is simply a function that respects the partial order. A *lattice* is a poset in which every pair of elements a, b has a greatest lower bound $a \wedge b$ (called their "meet"), and least upper bound $a \vee b$ (called their "join"). A lattice is *distributive* if these operations distribute across one another. A *lattice homomorphism* $L \to L'$ is a map of posets that preserves meets and joins.

One way to get a lattice is to take the set $\mathrm{Sub}(M)$ of submodules of any $\Bbbk$-module M; given $A, B \subseteq M$, we define $A \leq B$ iff $A \subseteq B$, $A \wedge B \coloneqq A \cap B$, and $A \vee B \coloneqq A + B$. This lattice is not, in general, distributive.

Definition 3.3.1. For a poset $\mathcal{P}$, a *$\mathcal{P}$-filtration* of a $\Bbbk$-module M is a map $\mathcal{F}_{-} : \mathcal{P} \to \mathrm{Sub}(M)$, i.e. a choice of submodule $\mathcal{F}_a M$ for each $a \in \mathcal{P}$ such that $\mathcal{F}_a M \subseteq \mathcal{F}_b M$ whenever $a \leq b$.

A filtration is *bounded* if there exists a finite interval $[s, t] \subseteq \mathcal{P}$ such that the restriction of $\mathcal{F}_{-}$ to $[s, t]$ is a map of bounded posets i.e. $\mathcal{F}_s = 0$ and $\mathcal{F}_t = M$. We will call a filtration *distributive* if the sublattice of $\mathrm{Sub}(M)$ generated by the image of $\mathcal{F}_{-}$ is distributive. These properties ensure that the associated graded objects are suitably nice. For this reason, we will assume that all $\mathcal{P}$-filtrations are bounded and distributive for the rest of the paper.

A *map of $\mathcal{P}$-filtered modules* is a linear map $f : M \to N$ such that $f(\mathcal{F}_a M) \subseteq \mathcal{F}_a(N)$ for all $a \in \mathcal{P}$. We will call a map of $\mathcal{P}$-filtered modules *distributive* if the sublattice of $\mathrm{Sub}(N)$ generated by $f(\mathcal{F}_a M)$ and $\mathcal{F}_a N$ for all $a \in \mathcal{P}$ is distributive. Again, we will assume that all maps of filtered modules are distributive for the rest of the paper.

Almost all of our constructions and definitions from Appendix A.1 can be generalized to posets, i.e. filtered isomorphisms, the induced and quotient filtration, and so on. The only notable exception is the associated graded object, which we will deal with shortly.

Associated graded objects. Given a $\mathcal{P}$-filtered module M, we define the *associated graded module* $\mathrm{gr}(M)$ to be the module

$$\mathrm{gr}(M) := \bigoplus_{a \in \mathcal{P}} \mathrm{gr}_a(M)$$
$$\mathrm{gr}_a(M) := \frac{\mathcal{F}_a M}{\sum_{b<a} \mathcal{F}_b M}$$

Here, the $\sum$ in the denominator refers to the sum of $\mathcal{F}_b M$ as submodules of $\mathcal{F}_a M$. One benefit of our definition of $\mathcal{P}$-filtered module is that it ensures that the associated graded is not "too big":

Theorem 3.3.2. $\mathrm{rank}\, M = \mathrm{rank}\, gr(M)$.

In proving this theorem, it will help to introduce a bit more notation. Let

$$\mathrm{gr}_{\leq a}(M) := \bigoplus_{b \leq a} \mathrm{gr}_b(M)$$

It turns out that the rank of the above module satisfies some nice properties, which we will encode in the following definition. A *valuation* on a bounded lattice L (see [27]) is a function $\mu : L \to \mathbb{R}$ such that

- $\mu(a \vee b) = \mu(a) + \mu(b) - \mu(a \wedge b)$
- $\mu(\perp) = 0$

When L is distributive, we can iterate the above conditions to obtain the *inclusion-exclusion principle*:

$$\mu(a_1 \vee a_2 \vee \ldots \vee a_n) = \sum_i \mu(a_i) - \sum_{i<j} \mu(a_i \wedge a_j) + \sum_{i<j<k} \mu(a_i \wedge a_j \wedge a_k) - \ldots$$

Essentially, this tells us that, in a distributive lattice, the valuation of a join of elements is entirely determined by the valuations of the meets of the elements. We will use this notion to prove the next lemma.

Lemma 3.3.3. $\operatorname{rank} gr_{\leq a}(M) = \operatorname{rank} \mathcal{F}_a(M)$.

Proof of Lemma 3.3.3. First, we will view both sides of the equation as valuations on the lattice L generated by the image of $\mathcal{F}_{-}$. Let $u(a) := \operatorname{rank} \mathrm{gr}_{\leq a}(M)$, and let

$v(a) \coloneqq \operatorname{rank} \mathcal{F}_a M$. We can see that both u and v are valuations, since

$$\begin{aligned}
u(\bot) &= \operatorname{rank} \operatorname{gr}_{\leq \bot}(M) \\
&= \operatorname{rank} \operatorname{gr}_{\bot}(M) \\
&= 0 \\
v(\bot) &= \operatorname{rank} \mathcal{F}_{\bot} M \\
&= 0 \\
u(a \vee b) &= \operatorname{rank} \operatorname{gr}_{\leq a \vee b}(M) \\
&= \operatorname{rank} \operatorname{gr}_{\leq a}(M) + \operatorname{rank} \operatorname{gr}_{\leq b}(M) - \operatorname{rank} \operatorname{gr}_{\leq a \wedge b}(M) \\
&= u(a) + u(b) - u(a \wedge b) \\
v(a \vee b) &= \operatorname{rank} \mathcal{F}_{a \vee b} M \\
&= \operatorname{rank}(\mathcal{F}_a M + \mathcal{F}_b M) \\
&= \operatorname{rank} \mathcal{F}_a M + \operatorname{rank} \mathcal{F}_b M - \operatorname{rank}(\mathcal{F}_a M \cap \mathcal{F}_b M) \\
&= \operatorname{rank} \mathcal{F}_a M + \operatorname{rank} \mathcal{F}_b M - \operatorname{rank} \mathcal{F}_{a \wedge b} M \\
&= v(a) + v(b) - v(a \wedge b)
\end{aligned}$$

With this in mind, we would like to prove that u and v are actually the *same* valuation on $\operatorname{Lower}(\mathcal{P})$.

We will proceed by induction. Our base case is already done for us, as $u(\bot) = 0 = v(\bot)$. For the inductive case, fix $a \in L$ and assume that we have proven the claim for all $b \in L$ such that $b < a$. Then

$$
\begin{aligned}
u(a) &= \operatorname{rank} \operatorname{gr}_{\leq a}(M) \\
&= \operatorname{rank} \bigoplus_{b \leq a} \operatorname{gr}_b(M) \\
&= \sum_{b \leq a} \operatorname{rank} \operatorname{gr}_b(M) \\
&= \operatorname{rank} \operatorname{gr}_a(M) + \sum_{b<a} \operatorname{rank} \operatorname{gr}_b(M) \\
&= \operatorname{rank} \operatorname{gr}_a(M) + u\left(\bigvee_{b<a} b\right) \\
&= \operatorname{rank} \operatorname{gr}_a(M) + v\left(\bigvee_{b<a} b\right) && \text{(by inclusion-exclusion)} \\
&= \operatorname{rank} \operatorname{gr}_a(M) + \operatorname{rank} \sum_{b<a} \mathcal{F}_b M \\
&= \operatorname{rank} \mathcal{F}_a M && \text{(by the definition of } \operatorname{gr}_a(M)\text{)} \\
&= v(a)
\end{aligned}
$$

□

Proof of Theorem 3.3.2. This follows from Lemma 3.3.3 with $a = \top \in \mathcal{P}$. □

$\mathcal{P}$-filtered complexes. As in the $\mathbb{Z}$-filtered case, a *$\mathcal{P}$-filtration of a chain complex* (C, ∂) is a $\mathcal{P}$-filtration of C that respects ∂ in the sense that $\partial(\mathcal{F}_a C) \subseteq \mathcal{F}_a C$ for $a \in \mathcal{P}$. A *$\mathcal{P}$-filtered chain map* is a map of chain complexes that also respects the filtration, i.e. a map of modules $f : C \to D$ that commutes with ∂ and "commutes" with $\mathcal{F}$. Similarly, the *associated graded complex* $\operatorname{gr}(C, \partial, \mathcal{F}) = (\operatorname{gr}(C), \operatorname{gr}(\partial))$ has underlying module the associated graded module and differential induced by the quotient operation. A *$\mathcal{P}$-filtered quasi-isomorphism* is a filtered chain map $f : C \to D$ that induces quasi-isomorphisms $\operatorname{gr}_a(f) : \operatorname{gr}_a(C) \to \operatorname{gr}_a(D)$ for all $a \in \mathcal{P}$.

We define strictness for $\mathcal{P}$-filtered maps the same way as we did for $\mathbb{Z}$-filtrations:

$f : C \to D$ is strict iff $f(\mathcal{F}_a C) = f(C) \cap \mathcal{F}_a D$ for all $a \in \mathcal{P}$. Since it relies on our new definition of associated graded, we will generalize the proof of Lemma A.1.1 from [51, Section 0120] to the $\mathcal{P}$-filtered case.

Lemma 3.3.4. *Let A be a $\mathcal{P}$-filtered complex, and let $X \subseteq A$ be a filtered subcomplex. Then*

$$0 \longrightarrow gr(X) \longrightarrow gr(A) \longrightarrow gr(A/X) \longrightarrow 0$$

is exact.

Proof. First, we rewrite $\mathrm{gr}_a(X)$ as

$$\mathrm{gr}_a(X) = \frac{\mathcal{F}_a X}{\sum_{b<a} \mathcal{F}_b X} = \frac{X \cap \mathcal{F}_a A}{\sum_{b<a} X \cap \mathcal{F}_b A} = \frac{X \cap \mathcal{F}_a A}{X \cap \sum_{b<a} \mathcal{F}_b A} \cong \frac{X \cap \mathcal{F}_a A + \sum_{b<a} \mathcal{F}_b A}{\sum_{b<a} \mathcal{F}_b A}$$

to see that the induced map to

$$\mathrm{gr}_a(A) = \frac{\mathcal{F}_a A}{\sum_{b<a} \mathcal{F}_b A}$$

is injective. Similarly, letting $\pi : A \to A/X$, we rewrite $\mathrm{gr}_a(A/X)$ as

$$\mathrm{gr}_a(A/X) = \frac{\mathcal{F}_a A/X}{\sum_{b<a} \mathcal{F}_b A/X} = \frac{\pi(\mathcal{F}_a A)}{\sum_{b<a} \pi(\mathcal{F}_b A)} = \frac{\pi(\mathcal{F}_a A)}{\pi(\sum_{b<a} \mathcal{F}_b A)}$$

$$= \frac{\dfrac{\mathcal{F}_a A}{X \cap \mathcal{F}_a A}}{\dfrac{\sum_{b<a} \mathcal{F}_b A}{X \cap \sum_{b<a} \mathcal{F}_b A}} = \frac{\dfrac{\mathcal{F}_a A}{X \cap \mathcal{F}_a A}}{\dfrac{\sum_{b<a} \mathcal{F}_b A}{X \cap \mathcal{F}_a A \cap \sum_{b<a} \mathcal{F}_b A}}$$

$$\cong \frac{\dfrac{\mathcal{F}_a A}{X \cap \mathcal{F}_a A}}{\dfrac{X \cap \mathcal{F}_a A + \sum_{b<a} \mathcal{F}_b A}{X \cap \mathcal{F}_a A}} \cong \frac{\mathcal{F}_a A}{X \cap \mathcal{F}_a A + \sum_{b<a} \mathcal{F}_b A}$$

to see that the induced map $\mathrm{gr}_a(A) \to \mathrm{gr}_a(A/X)$ is surjective. Thus, all we have left to do is check exactness in the middle. We have actually already done the work for this part, though; checking the numerator of $\mathrm{gr}_a(X)$ and the denominator of $\mathrm{gr}_a(A/X)$, we see that $\mathrm{im}(\mathrm{gr}_a(X) \to \mathrm{gr}_a(A))$ and $\ker(\mathrm{gr}_a(A) \to \mathrm{gr}_a(A/X))$ are equal, thus the sequence is exact. □

Lemma 3.3.5. *If $f : C \to D$ is a strict map of $\mathcal{P}$-filtered complexes, then the following sequence is exact:*

$$0 \longrightarrow gr(\ker f) \longrightarrow gr(C) \xrightarrow{gr(f)} gr(D) \longrightarrow gr(\operatorname{coker} f) \longrightarrow 0$$

Proof. First, we get that

$$0 \longrightarrow \mathrm{gr}(\ker f) \longrightarrow \mathrm{gr}(C) \longrightarrow \mathrm{gr}(\mathrm{coim}(f)) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{gr}(\mathrm{im}\, f) \longrightarrow \mathrm{gr}(D) \longrightarrow \mathrm{gr}(\mathrm{coker}(f)) \longrightarrow 0$$

are exact as special cases of Lemma 3.3.4. Here, $\mathrm{coim}(f) = C/\ker(f)$ is the coimage of f. We can stitch these two sequences together if we can show that $\mathrm{gr}(\mathrm{coim}(f)) \to \mathrm{gr}(\mathrm{im}(f))$ is an isomorphism.

To accomplish this, we will show that if f is strict, then the map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is a filtered isomorphism. This map is always an isomorphism of chain complexes, so it suffices to check that the map respects the filtrations. On $\mathrm{coim}(f)$, we have the quotient filtration from $C \twoheadrightarrow \mathrm{coim}(f)$, and on $\mathrm{im}(f)$ we have the induced filtration from $\mathrm{im}(f) \hookrightarrow D$. Since f is strict, we get that these filtrations coincide, i.e. for all $a \in \mathcal{P}$,

$$f(\mathcal{F}_a \,\mathrm{coim}(f)) = f(\mathcal{F}_a C) = f(C) \cap \mathcal{F}_a D = \mathcal{F}_a \,\mathrm{im}(f)$$

By functoriality, we then get that $\mathrm{gr}(\mathrm{coim}(f)) \to \mathrm{gr}(\mathrm{im}(f))$ is an isomorphism of graded complexes. Therefore, we get that

$$0 \longrightarrow \mathrm{gr}(\ker f) \longrightarrow \mathrm{gr}(C) \xrightarrow{\quad \mathrm{gr}(f) \quad} \mathrm{gr}(D) \longrightarrow \mathrm{gr}(\mathrm{coker}\, f) \longrightarrow 0$$
$$\mathrm{gr}(C) \searrow \mathrm{gr}(\mathrm{coim}(f)) \cong \mathrm{gr}(\mathrm{im}(f)) \nearrow \mathrm{gr}(D)$$

is exact. □

$\mathcal{P}$-filtered vector spaces. $\mathcal{P}$-filtered vector spaces have a simpler characterization. Let $\Bbbk$ be a field, and let V be a $\Bbbk$-vector space. As we have previously mentioned, the image of $\mathcal{F}_{-}$ in $\mathrm{Sub}(V)$ generates a finite distributive lattice. It turns out that such a lattice is isomorphic to a lattice of subsets, with operations given by set intersection and union. Specifically, V decomposes as a direct sum of subspaces V_a such that $\mathcal{F}_a V = \bigoplus_{b \leq a} V_a$ [43]. Equivalently, there exists a basis β of V such that each $\mathcal{F}_a V$ is the span of a subset of β. Therefore, we could instead specify a $\mathcal{P}$-filtration of V by fixing a basis β of V and giving a map from $\mathcal{P}$ to $\mathrm{Sub}(\beta)$ (the poset of subsets of β).

This is equivalent to specifying a grading of V by $\mathcal{P}$, which is the approach we will take in Section 3.4 and onward.

Section 3.4

Multipunctured Khovanov homology

In this section, we define the main construction of this paper. We start by defining a $\mathbb{Z}^n$-grading on (the underlying module of) the Khovanov complex of a link diagram in an n-punctured disk, then use it to build a homology theory in a way that is analogous to annular Khovanov homology.

Let $\mathbb{I} = [0,1]$ be the unit interval, let $\Sigma = \mathbb{D}^2 \setminus \{p_1, \ldots, p_n\}$ be an oriented disk with n punctures, and let $L \subset \Sigma \times \mathbb{I}$ be any link (considered up to ambient isotopy). Let $D \subset \Sigma$ denote the link diagram obtained by projecting L onto Σ (with small perturbations to remove triple points and tangencies if necessary), and let D be the link diagram in $\mathbb{R}^2$ induced by the inclusion $\Sigma \hookrightarrow \mathbb{R}^2$.

After picking an ordering of the crossings in D, we can construct the Khovanov complex $\mathrm{CKh}(D)$ as in Section 2.2. Let $\Bbbk = \mathbb{Q}$ be our coefficient ring. We would like to construct an $H_1(\Sigma;\mathbb{Z})$-grading g^Σ on the module $\mathrm{CKh}(D)$ that "remembers" the extra structure of $D \subset \Sigma$; it suffices to specify such a grading on the generators. Let $x \in \mathrm{CKh}(D)$ be a generator represented by the enhanced state $\{(C_i, s_i)\}_i$; the $H_1(\Sigma;\mathbb{Z})$-grading of x is then defined to be

$$\mathrm{g}^\Sigma(x) := \sum_{i=1}^{j} s_i\,[C_i] \qquad \in H_1(\Sigma;\mathbb{Z}) \tag{3.4.1}$$

where $[C_i]$ is the homology class represented by C_i in Σ.

Note that the differential on $\mathrm{CKh}(D)$ does not respect the g^Σ. We claim that the differential on $\mathrm{CKh}(D)$ does, however, respect the filtration $\mathcal{F}^\Sigma$ induced by g^Σ, if we view $H_1(\Sigma;\mathbb{Z})$ as a poset in a certain way. First, choose the basis of $H_1(\Sigma;\mathbb{Z})$ represented by positively-oriented loops around each puncture. We use this basis to identify $H_1(\Sigma;\mathbb{Z})$ with $\mathbb{Z}^n$. Then, give it the product partial order induced by the usual total order on $\mathbb{Z}$, i.e.

$$(i_1, i_2, \dots, i_n) \leq (j_1, j_2, \dots, j_n) \iff (i_1 \leq j_1) \wedge (i_2 \leq j_2) \wedge \cdots \wedge (i_n \leq j_n)$$

Our claim is thus:

Lemma 3.4.1. *The Khovanov differential ∂ respects the filtration $\mathcal{F}^\Sigma$ on* $\mathrm{CKh}(D)$*.*

Proof. We want to show that the differential and filtration on $(\mathrm{CKh}(D), \partial, \mathcal{F}^\Sigma)$ are compatible, i.e. that $\partial(\mathcal{F}^\Sigma_p \mathrm{CKh}(D)) \subseteq \mathcal{F}^\Sigma_p \mathrm{CKh}(D)$. Thankfully, this proof comes almost for free from annular Khovanov homology. Choose a puncture p_i and consider $D \subset \Sigma$ as a diagram in $\Sigma' = \mathbb{D}^2 \setminus \{p_i\}$; essentially, we are "forgetting" all but one of of the punctures. Let $\mathcal{F}'$ be the filtration on $\mathrm{CKh}(D \subset \Sigma')$. The diagram $D \subset \Sigma'$ is an annular link diagram; in [50], it was shown that the Khovanov differential is compatible with a certain filtration on the Khovanov complex of an annular link. When the diagram $D \subset \Sigma'$ only has one puncture, the filtration induced by our grading g^Σ is equivalent to their filtration, so $\partial(\mathcal{F}'_p \mathrm{CKh}(D)) \subseteq \mathcal{F}'_p \mathrm{CKh}(D)$. Since this is true regardless of which puncture p_i we pick, this implies that $\partial(\mathcal{F}^\Sigma_p \mathrm{CKh}(D)) \subseteq \mathcal{F}^\Sigma_p \mathrm{CKh}(D)$. Thus, the filtration on $\mathrm{CKh}(D \subset \Sigma)$ is compatible with the chain complex structure. □

Note that, since the complex $\mathrm{CKh}(D)$ is finitely-generated over $\mathbb{Q}$, the set of g^Σ-gradings of generators of $\mathrm{CKh}(D)$ is finite, and in particular bounded. Therefore, there are elements $s, t \in \mathbb{Z}^n$ such that $s < \mathrm{g}^\Sigma(x) < t$ for all generators x of $\mathrm{CKh}(D)$. This means that $\mathcal{F}^\Sigma_s = 0$ and $\mathcal{F}^\Sigma_t = \mathrm{CKh}(D)$, so our filtration is bounded. Additionally, since our filtration is induced by a grading, by Section 3.3.1 it is distributive as well.

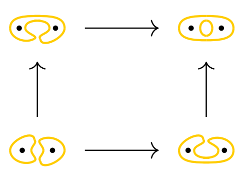
Now that we know that $\mathcal{F}^\Sigma$ is a bounded, distributive filtration on the chain complex $\mathrm{CKh}(D)$, we can define our invariant.

Definition 3.4.2. The link invariant $\mathrm{MKh}(L)$[2] is defined to be the homology of the associated graded complex of $\mathrm{CKh}(D)$ with respect to $\mathcal{F}^\Sigma$.

$$\mathrm{MKh}(L) \coloneqq H^*(\mathrm{gr}(\mathrm{CKh}(D), \partial, \mathcal{F}^\Sigma))$$

That the above is actually a well-defined link invariant is the content of Theorem 3.1.1, which we prove in Section 3.7.

Example 3.4.3. Let $L =$ . Here, the black dots represent punctures. We again construct a cube of resolutions:



$$\begin{array}{ccc} V & \xrightarrow{\Delta_2} & V^{\otimes 2} \\ {\scriptstyle m_1}\uparrow & & \uparrow{\scriptstyle \Delta_1} \\ V^{\otimes 2} & \xrightarrow{m_2} & V \end{array}$$

The differential can be written explicitly as:

$$\mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 1&0&0&0\\ 0&1&1&0\\ 1&0&0&0\\ 0&1&1&0 \end{pmatrix}} \mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 0&0&0&0\\ -1&0&1&0\\ -1&0&1&0\\ 0&-1&0&1 \end{pmatrix}} \mathbb{Q}^4$$

Next, we calculate the differential on the associated graded complex:

$$\mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}} \mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 0&0&0&0\\ -1&0&1&0\\ 0&0&0&0\\ 0&-1&0&1 \end{pmatrix}} \mathbb{Q}^4$$

As in the annular case, this can be thought of as removing all parts of the differential that lower the g^Σ-grading. For example, in the original complex, we have that

$\partial($  $) =$  $+$ 

v_+ $\qquad$ $v_+ \otimes v_-$ $\qquad$ $v_- \otimes v_+$

[2]We chose MKh for lack of a better abbreviation. The "M" stands for "multipunctured", not "Mikhail".

but in the associated graded complex, we have

$\partial($  $) =$ 

v_+ $\qquad$ $v_+ \otimes v_-$

We can calculate that:

$$\begin{aligned} \mathrm{MKh}^{-2}(L) &\cong \mathbb{Q}^2\{-4\} \oplus \mathbb{Q}\{-6\} \\ \mathrm{MKh}^{-1}(L) &\cong \mathbb{Q}\{-4\} \\ \mathrm{MKh}^{0}(L) &\cong \mathbb{Q} \oplus \mathbb{Q}\{-2\} \end{aligned}$$

Using subscripts to denote the g^{Σ}-grading, we can refine this further to:

$$\begin{aligned} \mathrm{MKh}^{-2}(L) &\cong \mathbb{Q}_{(1,-1)}\{-4\} \oplus \mathbb{Q}_{(-1,1)}\{-4\} \oplus \mathbb{Q}_{(-1,-1)}\{-6\} \\ \mathrm{MKh}^{-1}(L) &\cong \mathbb{Q}_{(-1,-1)}\{-4\} \\ \mathrm{MKh}^{0}(L) &\cong \mathbb{Q}_{(1,1)} \oplus \mathbb{Q}_{(-1,-1)}\{-2\} \end{aligned}$$

Section 3.5

Spectral sequences

One notable fact about annular Khovanov homology is that, given a link L, there is a spectral sequence with E_1 page isomorphic to $\mathrm{AKh}(L)$ that converges to $\mathrm{Kh}(L)$. It is natural to ask if our homology fits into similar spectral sequences as well.

Let Σ be an n-punctured disk, let L be a link in Σ, and let Σ' be a disk with some subset of $m \leq n$ of these punctures. We can view L as a link in Σ' by the natural inclusion map $\Sigma \hookrightarrow \Sigma'$. Note that $\mathrm{MKh}(L \subset \Sigma)$ is naturally a $H_1(\Sigma;\mathbb{Z})$-graded module, and $\mathrm{MKh}(L \subset \Sigma')$ is likewise graded by $H_1(\Sigma';\mathbb{Z})$. In order to use the classical results about spectral sequences to compare these two, we will first turn them into $\mathbb{Z}$-graded modules. Let $\{v_1, v_2, \ldots, v_n\}$ be the basis of $H_1(\Sigma)$ where v_i corresponds to a loop around the i-th puncture. Let $\epsilon : H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}$ be the map of posets that sends an element to the sum of its components:

$$\epsilon : \sum_{1 \leq i \leq n} c_i v_i \mapsto \sum_{1 \leq i \leq n} c_i \tag{3.5.1}$$

Let D be a diagram for L. We have a $H_1(\Sigma; \mathbb{Z})$-grading g^Σ defined on generators of $\mathrm{CKh}(D)$ above in Equation (3.4.1); taking the composition $\epsilon \circ \mathrm{g}^\Sigma$ gives us a $\mathbb{Z}$-grading. Since the grading g^Σ induced a filtration on $\mathrm{CKh}(D)$ as a chain complex, and ϵ is a map of posets, we have that $\epsilon \circ \mathrm{g}^\Sigma$ induces a $\mathbb{Z}$-filtration on $\mathrm{CKh}(D)$. We will denote this filtration by F^Σ (a flatter $\mathcal{F}$ for a "flattened" filtration). Similarly, we can define a map $\epsilon' : H_1(\Sigma'; \mathbb{Z}) \to \mathbb{Z}$ and a corresponding induced filtration $F^{\Sigma'}$ on $\mathrm{CKh}(D)$.

First, we need a lemma to help us compare these two filtrations on $\mathrm{CKh}(D)$:

Lemma 3.5.1. *Let $\mathcal{F}^1, \mathcal{F}^2$ be two $\mathcal{P}$-filtrations on a module M. Then $\mathcal{F}^1$ induces a $\mathcal{P}$-filtration on the associated graded module of M with respect to $\mathcal{F}^2$. Additionally, if the sublattice of* $\mathrm{Sub}(M)$ *generated by $\mathcal{F}^1$ and $\mathcal{F}^2$ is distributive, then this new filtration is also distributive.*

Proof. First, we describe the filtration that $\mathcal{F}^1_-$ induces on $\mathcal{F}^2_b M$:

$$\mathcal{F}^1_a(\mathcal{F}^2_b M) \coloneqq \mathcal{F}^1_a M \cap \mathcal{F}^2_b M$$

Let $\pi : \mathcal{F}^2_b M \to \mathrm{gr}^2_b M$ be the natural quotient map. We can use π to transfer the induced filtration onto the associated graded object:

$$\mathcal{F}^1_p(\mathrm{gr}^2_q M) \coloneqq \pi(\mathcal{F}^1_p(\mathcal{F}^2_q M)) = \frac{\mathcal{F}^1_p(\mathcal{F}^2_q M) + \mathcal{F}^2_{q-1} M}{\mathcal{F}^2_{q-1} M}$$

We can then take the direct sum of each $\mathrm{gr}_b(M)$ to get the desired filtration on $\mathrm{gr}(M)$.

If $\mathcal{F}^1$ and $\mathcal{F}^2$ distribute, then

$$- \cap \mathcal{F}^2_b : \mathrm{Sub}(M) \to \mathrm{Sub}(\mathcal{F}^2_b)$$

and

$$\mathrm{Sub}(\pi) : \mathrm{Sub}(\mathcal{F}^2_b M) \to \mathrm{Sub}(\mathrm{gr}_b(M))$$

are lattice homomorphisms, and therefore preserve distributive sublattices. Therefore, $\mathcal{F}^1_- \mathrm{gr}_b(M) = \mathrm{Sub}(\pi)(\mathcal{F}^1_- M \cap \mathcal{F}^2_b M)$ will be distributive as well. □

The above lemma also applies to modules with extra structure, for example chain complexes. It turns out that, if two filtrations on a chain complex are suitably "compatible", we get a spectral sequence between the homologies of their associated graded complexes:

Lemma 3.5.2. *Let $\mathcal{F}^1, \mathcal{F}^2$ be two $\mathbb{Z}$-filtrations on a chain complex C, and let gr^1, gr^2 denote their associated graded complexes. If $gr^1(gr^2(C)) \cong gr^1(C)$ as graded complexes, then there is a spectral sequence with E_1 page isomorphic to $H_*(gr^1(C))$ that converges to $gr^1(H_*(gr^2(C)))$.*

Proof. Consider the chain complex $\mathrm{gr}^2(C)$ with filtration induced by $\mathcal{F}^1$. The spectral sequence of this filtered complex has E_1 page isomorphic to

$$E_1 \cong H_*(\mathrm{gr}^1(\mathrm{gr}^2(C))) \cong H_*(\mathrm{gr}^1(C))$$

Additionally, the spectral sequence converges to

$$E_\infty \cong \mathrm{gr}^1(H_*(\mathrm{gr}^2(C)))$$ □

Therefore, if we could prove that $\mathrm{gr}^{\Sigma}(\mathrm{gr}^{\Sigma'}(C)) \cong \mathrm{gr}^{\Sigma}(C)$, then we would have a spectral sequence relating our two homologies. This last lemma will help with proving that:

Lemma 3.5.3. *Let $\mathcal{F}^1, \mathcal{F}^2$ be two filtrations on a chain complex (C, ∂) over a field $\Bbbk$, and let $\mathrm{g}^1, \mathrm{g}^2$ be their respective gradings on C. If, for any $x, y \in C$ with y contained in $\partial(x)$, we have that $\mathrm{g}^1(y) = \mathrm{g}^1(x) \implies \mathrm{g}^2(y) = \mathrm{g}^2(x)$, then there is an isomorphism of $\mathbb{Z}$-graded chain complexes $gr^1(gr^2(C)) \cong gr^1(C)$.*

Proof. Consider C as a $(\mathrm{g}^1, \mathrm{g}^2)$-bigraded module. We already know that $\mathrm{gr}(M) \cong M$ for any filtered vector space M, so we have that $\mathrm{gr}^1(\mathrm{gr}^2(C)) \cong \mathrm{gr}^1(\mathrm{CKh}(D))$ as modules. The interesting part is proving the analogous fact in the presence of a differential.

Our bigrading allows us to decompose ∂ as:

$$\partial = \sum_{i,j} \partial^{(i,j)}$$

where $\partial^{(i.j)}$ is the summand of ∂ that changes the g^1-grading by i and the g^2-grading by j. We can use this notation to express the differentials on the associated graded complexes as well:

$$\mathrm{gr}^1(\partial) = \sum_j \partial^{(0,j)} = \partial^{(0,0)}$$
$$\mathrm{gr}^2(\partial) = \sum_i \partial^{(i,0)} = \sum_i \partial^{(i,0)}$$

Therefore, we see that $\mathrm{gr}^1(\partial)$ is a summand of $\mathrm{gr}^2(\partial)$, so

$$\mathrm{gr}^1(\mathrm{gr}^2(\partial)) = \partial^{(0,0)} = \mathrm{gr}^1(\partial)$$

This allows us to conclude that $\mathrm{gr}^1(\mathrm{gr}^2(\mathrm{CKh}(D))) \cong \mathrm{gr}^1(\mathrm{CKh}(D))$ as $\mathbb{Z}$-graded chain complexes with respect to the g^1-grading. □

With this in mind, we can now relate $\mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$ and $\mathrm{MKh}(L \subset \Sigma' \times \mathbb{I})$.

Proof of Theorem 3.1.6. We would like to verify that F^{Σ} and $F^{\Sigma'}$ satisfy the conditions of Lemma 3.5.3. Recall that, with respect to a single puncture, the Khovanov differential either preserves the k-grading (equivalently, the g^A-grading), or lowers it by 2. Therefore, ∂ can lower the $(\epsilon \circ \mathrm{g}^{\Sigma})$-grading by $0, 2, 4, \dots,$ and can lower the $(\epsilon \circ \mathrm{g}^{\Sigma'})$-grading by at most that amount. If $x, y \in \mathrm{CKh}(D)$ are such that y is contained in $\partial(x)$, we have that

$$(\epsilon \circ \mathrm{g}^{\Sigma})(y) - (\epsilon \circ \mathrm{g}^{\Sigma})(x) \leq (\epsilon \circ \mathrm{g}^{\Sigma'})(y) - (\epsilon \circ \mathrm{g}^{\Sigma'})(x) \leq 0$$

and therefore

$$(\epsilon \circ \mathrm{g}^{\Sigma})(y) - (\epsilon \circ \mathrm{g}^{\Sigma})(x) = 0 \implies (\epsilon \circ \mathrm{g}^{\Sigma'})(y) - (\epsilon \circ \mathrm{g}^{\Sigma'})(x) = 0$$

Now we can use Lemma 3.5.3 combined with Lemma 3.5.2 to conclude that there exists a spectral sequence with

$$E_1 \cong H^*(\mathrm{gr}^{\Sigma}(\mathrm{CKh}(D))) \cong \mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$$

that converges to

$$E_\infty \cong \mathrm{gr}^{\Sigma} H^*(\mathrm{gr}^{\Sigma'}(\mathrm{CKh}(D))) \cong \mathrm{MKh}(L \subset \Sigma' \times \mathbb{I})$$

where both homologies are graded by $\epsilon \circ \mathrm{g}^{\Sigma}$. □

Section 3.6

Relationships with other link homology theories

3.6.1. (Annular) Khovanov homology

We can use the spectral sequences defined in Section 3.5 to draw a few different connections to Khovanov homology and annular Khovanov homology. First, we identify special cases where $\mathrm{MKh}(L)$ is already isomorphic to a previously-defined link invariant.

Lemma 3.6.1. *Let $\mathbb{D}$ be a disk, and let D be a diagram for a link $L \subset \mathbb{D} \times \mathbb{I}$. Then $\mathrm{MKh}(L \subset \mathbb{D} \times \mathbb{I}) \cong \mathrm{Kh}(L)$.*

Proof. Since $H_1(\mathbb{D}; \mathbb{Z}) \cong 0$, $\mathrm{g}^{\mathbb{D}}$ is a trivial grading, and therefore

$$\mathrm{MKh}(L \subset \mathbb{D} \times \mathbb{I}) = H^*(\mathrm{gr}^{\mathbb{D}}(\mathrm{CKh}(D))) \cong H^*(\mathrm{CKh}(D)) = \mathrm{Kh}(L) \qquad \square$$

Lemma 3.6.2. *Let A be an annulus, and let $L \subset A \times \mathbb{I}$ be an annular link. Then $\mathrm{MKh}(L \subset A \times \mathbb{I}) \cong \mathrm{AKh}(L)$.*

Proof. Since $H_1(A; \mathbb{Z}) \cong \mathbb{Z}$, we get that the k- and g^A-gradings are identical. Therefore, for a diagram D of L:

$$\mathrm{MKh}(L \subset A \times \mathbb{I}) = H^*(\mathrm{gr}^A(\mathrm{CKh}(D)) \cong H^*(\mathrm{gr}^k(\mathrm{CKh}(D))) = \mathrm{AKh}(D) \qquad \square$$

As a first application of Theorem 3.1.6, we get a spectral sequence from $\mathrm{MKh}(L)$ to $\mathrm{Kh}(L)$:

Proof of Theorem 3.1.5. This follows immediately from Lemma 3.6.1 as a special case of Theorem 3.1.6 with $\Sigma = \Sigma$ and $\Sigma' = \mathbb{D}$ a disk. $\square$

Additionally, this lets us derive the known spectral sequence from annular Khovanov homology to Khovanov homology as a special case of our construction.

Corollary 3.6.3. *For any annular link L, there is a spectral spectral sequence with E_1 page isomorphic to* $\mathrm{AKh}(L)$ *that converges to* $\mathrm{Kh}(L)$.

Proof. This follows immediately from Lemma 3.6.2 as a special case of Theorem 3.1.6 with $\Sigma = A$ an annulus and $\Sigma' = \mathbb{D}$ a disk. □

We get a second connection to annular Khovanov homology when Σ is a disk with multiple punctures, and we single out a particular puncture.

Proof of Theorem 3.1.4. The proof follows from Lemma 3.6.2 as a special case of Theorem 3.1.6 with $\Sigma = \Sigma$ and $\Sigma' = A$. □

3.6.2. APS homology

"APS homology" is the name we will use to refer to the invariant of links in thickened $\mathbb{I}$-bundles as defined in [2]; we will denote it $\mathrm{APS}(L)$. Annular Khovanov homology originated as a specialization of this theory; since our homology is a generalization of annular Khovanov homology, one might expect there to be connections between our homology and APS homology.

As originally defined, $\mathrm{APS}(L)$ is an invariant of framed links, and therefore is not invariant under Reidemeister I moves. However, it is shown in [2, Theorem 6.2] that a change in framing induces only a shift in the gradings of $\mathrm{APS}(L)$, and is therefore easily accounted for in defining a framing-independent invariant. We will denote this normalization by $\widetilde{\mathrm{APS}}(L)$, noting that it is isomorphic to APS homology as a module, but is an invariant of (unframed) links and follows the same grading conventions we have used thus far. For completeness, we now proceed to give a definition of $\widetilde{\mathrm{APS}(L)}$.

Let $D \subset \Sigma$ be a link diagram. Let $C(\Sigma)$ denote the set of all homotopy classes of unoriented, non-trivial simple closed curves in Σ. Let $x \in \mathrm{CKh}(D)$ be a generator representing a complete resolution of D with signed circles $\{(C_i, s_i)\}_i$. We define a grading on $\mathrm{CKh}(D)$ as:

$$\Phi(x) := \sum_i s_i \langle C_i \rangle \qquad \in \mathbb{Z}C(\Sigma)$$

Note that the definition of Φ is very similar to that of g^Σ, the difference being that Φ takes values in $ZC(\Sigma)$ instead of $H_1(\Sigma;\mathbb{Z})$. We would like to define $\widetilde{\mathrm{APS}}(L)$ as the homology of the associated graded complex of $\mathrm{CKh}(D)$ with respect to Φ; however, Φ does not induce a filtration on $\mathrm{CKh}(D)$ with the product partial order on $\mathbb{Z}C(\Sigma)$, so we need to work around this.

Let $\epsilon : \mathbb{Z}C(\Sigma) \to \mathbb{Z}$ be the map of posets defined by summing the components analogously to Equation (3.5.1), i.e.

$$\epsilon : \sum_i s_i \langle C_i \rangle \mapsto \sum_i s_i$$

Define a new partial order $\trianglelefteq$ on $\mathbb{Z}C(\Sigma)$, where $\alpha \trianglelefteq \beta$ if and only if either $\alpha = \beta$ or $\epsilon(\alpha) < \epsilon(\beta)$.

Lemma 3.6.4. *The Khovanov differential is filtered with respect to* $(\mathbb{Z}C(\Sigma), \trianglelefteq)$.

Proof. Let D be a link diagram in Σ, and let $x, y \in \mathrm{CKh}(D)$ be generators such that $\partial(x)$ contains a non-zero multiple of y as a summand. We have a few cases to consider:

- If x and y are related by a merge or split involving only non-trivial circles, then $\epsilon(\Phi(y)) = \epsilon(\Phi(x)) - 1$ since we know that ∂ preserves the q-grading. Therefore, $\epsilon(\Phi(y)) < \epsilon(\Phi(x))$.

- If y is obtained from x by splitting a trivial circle into two non-trivial circles, then it must be that both non-trivial circles represent the same class in $C(\Sigma)$.

Therefore, either $\Phi(x) = \Phi(y)$ (if the trivial circle is marked with a $+$) or $\epsilon(\Phi(y)) = \epsilon(\Phi(x)) - 2$ (if the trivial circle is marked with a $-$).

- If y is obtained from x by splitting a non-trivial circle into a trivial circle and a non-trivial circle, then both non-trivial circles represent the same class in $C(\Sigma)$, so $\Phi(x) = \Phi(y)$.

- If y is obtained from x by merging a non-trivial circle and a trivial circle, then the resulting circle is non-trivial, and represents the same class in $C(\Sigma)$, so $\Phi(x) = \Phi(y)$.

- If y is obtained from x by merging two non-trivial circles to obtain a trivial circle, then the two non-trivial circles must represent the same class in $C(\Sigma)$, so $\Phi(x) = \Phi(y)$.

- In all other cases, $\Phi(x) = \Phi(y)$. □

Now, we have the language needed to define $\widetilde{\mathrm{APS}}(L)$:

Definition 3.6.5. Let D be a diagram for a link L in $\Sigma \times \mathbb{I}$. Regard $\mathrm{CKh}(D)$ as a $\mathbb{Z}C(\Sigma)$-filtered module under the partial order $\trianglelefteq$. Define

$$\widetilde{\mathrm{APS}}(L) \coloneqq H^*(\mathrm{gr}^{\Phi}(\mathrm{CKh}(D)))$$

Proposition 3.6.6 (c.f. [2, Theorem 6.2])**.** $\widetilde{\mathrm{APS}}(L)$ *is an invariant of links in* $\Sigma \times \mathbb{I}$ *up to ambient isotopy.*

Proof. For $S = \{(C_i, s_i)\}_i$ a resolution of a diagram D, the original gradings on APS homology[3] are

- $I(S) \coloneqq \#\{0\text{-resolved crossings}\} - \#\{1\text{-resolved crossings}\}$

[3] The original definition used the opposite convention regarding $+$ and $-$ labels on circles, as well as different terminology used to describe resolutions.

- $J(S) \coloneqq I(S) + 2\tau(S)$,

  where $\tau(S) \coloneqq \#\{\text{trivial circles labeled } -\} - \#\{\text{trivial circles labeled } +\}$

- $\Psi(S) \coloneqq \sum_i (-s_i) \langle C_i \rangle$

  We can match these with our gradings on $\widetilde{\mathrm{APS}}(L)$ as follows:

- $\mathrm{gr}_h(S) = \frac{1}{2}(n - I(S))$

- $\mathrm{gr}_q(S) = -(\epsilon(\Psi(S)) + \tau(S)) + \mathrm{gr}_h(S) + \mathrm{wr}(L)$

- $\Phi(S) = -\Psi(S)$

In [2], it was proven that $\mathrm{APS}(L)$ is an invariant of framed links, with Reidemeister I moves inducing isomorphisms of modules with fixed grading shifts. With this change of gradings, $\widetilde{\mathrm{APS}}(L)$ is actually an invariant of unframed links. □

Fix an ordering on the basis of $\mathbb{Z}C(\Sigma)$, and consider the ring of Laurent polynomials $\mathbb{Z}[q^{\pm 1}, x_{c_1}^{\pm 1}, x_{c_2}^{\pm 1}, \dots]$ for $c_i \in C(\Sigma)$. When Σ is implied, we will denote the above using the shorthand $\mathbb{Z}[q^{\pm 1}, x^{\pm 1}]$. Additionally, given a vector $v = (v_1, v_2, \dots) \in \mathbb{Z}C(\Sigma)$, let

$$x^v \coloneqq \prod_i x_{c_i}^{v_i} = x_{c_1}^{v_1} x_{c_2}^{v_2} \dots$$

Similarly, we will write $\mathbb{Z}[q^{\pm 1}, y^{\pm 1}]$ to denote the ring $\mathbb{Z}[q^{\pm 1}, y_{c_1}^{\pm 1}, y_{c_2}^{\pm 1}, \dots, y_{c_n}^{\pm 1}]$ for a basis $\{c_i\}_{i=1}^n$ of $H_1(\Sigma;\mathbb{Z})$, and define y^v analogously for $v \in H_1(\Sigma;\mathbb{Z})$.

Definition 3.6.7. Define the Euler characteristics of our homologies as follows:

$$\chi(\widetilde{\mathrm{APS}}(L)) \coloneqq \sum_{i,j,v} (-1)^i q^j x^v \,\mathrm{rk}(\widetilde{\mathrm{APS}}^{i,j,v}(L)) \qquad \in \mathbb{Z}[q^{\pm 1}, x^{\pm 1}]$$

$$\chi(\mathrm{MKh}(L)) \coloneqq \sum_{i,j,v} (-1)^i q^j y^v \,\mathrm{rk}(H^{i,j,v}(L)) \qquad \in \mathbb{Z}[q^{\pm 1}, y^{\pm 1}]$$

Define $h : \mathbb{Z}C(\Sigma) \to H_1(\Sigma; \mathbb{Z})$ on generators as $h(c) = [c]$, and extend linearly. We can then see that h induces a map between these two Euler characteristics:

Proof of Theorem 3.1.2. Define the map χ_h by letting $\chi_h(q) = q$ and $\chi_h(x_c) = y_{[c]}$, then extending algebraically. The proof that $\chi_h(\chi(\widetilde{\mathrm{APS}}(L))) = \chi(\mathrm{MKh}(L))$ follows from the fact that both homologies come from filtrations on the same chain complex. □

We actually have a stronger connection between $\widetilde{\mathrm{APS}}(L)$ and $\mathrm{MKh}(L)$. In the same vein as Section 3.5, we can construct a spectral sequence between the two homologies. As before, we will "flatten" the filtrations so that we can use the standard spectral sequence construction[4].

For a link diagram $D \subset \Sigma$, let F^Φ be the $\mathbb{Z}$-filtration induced by the $\mathbb{Z}$-grading $\epsilon \circ \Phi$ on $\mathrm{CKh}(D)$. Similarly, let F^Σ be the $\mathbb{Z}$-filtration induced by the $\mathbb{Z}$-grading $\epsilon \circ \mathrm{g}^\Sigma$ on $\mathrm{CKh}(D)$. We can prove the analogous statement to Lemma 3.5.3 for these filtrations.

Lemma 3.6.8. *Let gr^Φ denote the associated graded object with respect to F^Φ, and define gr^Σ analogously. Then there is an isomorphism of $\mathbb{Z}$-graded chain complexes $gr^\Phi(gr^\Sigma \mathrm{CKh}(D)) \cong gr^\Phi \mathrm{CKh}(D)$.*

Proof. Our filtrations F^Φ and F^Σ come from two gradings Φ and g^Σ (respectively) on $\mathrm{CKh}(D)$, so we can consider $\mathrm{CKh}(D)$ as a $(\Phi, \mathrm{g}^\Sigma)$-bigraded module. We already know that $\mathrm{gr}(M) \cong M$ for any filtered vector space M, so we have that $\mathrm{gr}^\Phi(\mathrm{gr}^\Sigma(\mathrm{CKh}(D))) \cong \mathrm{gr}^\Phi(\mathrm{CKh}(D))$ as modules. The interesting part is proving the analogous fact for the differential.

Let ∂ denote the Khovanov differential on $\mathrm{CKh}(D)$. We would like to show that, for any generators $x, y \in \mathrm{CKh}(D)$ with y contained in $\partial(x)$, if $(\epsilon\circ\Psi)(y) - (\epsilon\circ\Psi)(x) = 0$,

[4] Morally, this should probably also be thought of as induced by h, but flattening the gradings obscures this relationship.

then $(\epsilon \circ \mathrm{g})(y) - (\epsilon \circ \mathrm{g})(x) = 0$. This requires checking the same cases as in the proof of Lemma 3.6.4 where $\Phi(x) = \Phi(y)$. It turns out that in every case, we get two circles with the same class in $C(\Sigma)$ cancelling each other out. Additionally, if C_1 and C_2 are two circles representing the same class in $C(\Sigma)$, then they must also represent the same class in $H_1(\Sigma)$. Therefore, we get that $(\epsilon \circ \mathrm{g})(y) - (\epsilon \circ \mathrm{g})(x) = 0$ as well.

Since $(\epsilon \circ \Psi)(y) - (\epsilon \circ \Psi)(x) = 0$ implies that $(\epsilon \circ \mathrm{g})(y) - (\epsilon \circ \mathrm{g})(x) = 0$, we get that $\mathrm{gr}(\mathrm{gr}' \, \mathrm{CKh}(D)) \cong \mathrm{gr}\, \mathrm{CKh}(D)$. □

Finally, we have all the necessary pieces together to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. Our spectral sequence is induced by the filtered complex $(\mathrm{gr}'(\mathrm{CKh}(D)), \mathcal{F})$. The E_0 page of this spectral sequence is $\mathrm{gr}(\mathrm{gr}' \, \mathrm{CKh}(D))$, which is isomorphic to $\mathrm{gr}(\mathrm{CKh}(D))$ as a chain complex by Lemma 3.6.8. Therefore, by Lemma 3.5.2, we have a spectral sequence with

$$E_1 \cong H^*(\mathrm{gr}(\mathrm{CKh}(D)) \cong \widetilde{\mathrm{APS}}(L)$$

that converges to the underlying module of

$$E_\infty \cong \mathrm{gr}\, H^*(\mathrm{gr}' \, \mathrm{CKh}(D)) \cong H^*(\mathrm{gr}' \, \mathrm{CKh}(D)) \cong \mathrm{MKh}(L \subset \Sigma \times \mathbb{I})$$ □

Section 3.7

Invariance

In this section, we finish what we started in Section 3.4 by providing the proofs that $\mathrm{MKh}(L)$ is a well-defined link invariant, thereby proving Theorem 3.1.1.

Remark 3.7.1. One nice feature about chain complexes arising from a cube of resolutions is that they can naturally be presented as complexes of complexes (of complexes of..., etc.) Throughout this section, we will often implicitly identify double complexes with their total complexes and vice versa. It is worth pointing out that this trick still works when dealing with filtered objects.

Specifically, we note that a chain complex of filtered modules is the same thing as a filtered chain complex. This allows us to think about chain complexes of filtered chain complexes as filtered double complexes, which we then flatten into filtered chain complexes.

3.7.1. Reidemeister I invariance

Remark 3.7.2. For the next three subsections, we aim to mimic Bar-Natan's argument for the invariance of Khovanov homology (see [11]), while checking some extra conditions to ensure that the our filtration on the Khovanov complex is also suitably invariant. For some diagram D, we will use the notation $[\![D]\!]$ throughout to denote the filtered chain complex $\mathrm{CKh}(D_\Sigma)$, and we will often describe complexes as complexes of complexes as in Remark 3.7.1.

Unfiltered version. We would like to show that the complexes $[\![$$]\!]$ and $[\![$$]\!]$ have the same homology. Let

$$C = [\![$$  $$]\!] = \Big([\![$$  $$]\!] \xrightarrow{m} [\![$$  $$]\!] \{1\} \Big)$$

and let

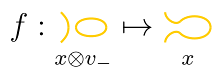
$$C' = \Big([\![$$  $$]\!]_{v_+} \xrightarrow{m} [\![$$  $$]\!] \{1\} \Big) \subseteq C$$

The notation $[\![\ldots]\!]_{v_+}$ denotes the subcomplex consisting of all generators represented by diagrams in which the loop pictured in the diagram is labeled with a $+$. Note that, since v_+ is a "unit" for the merge m, it follows that m is an isomorphism in C', so C'

is acyclic. This means that the quotient map $q : C \to C/C'$ is a quasi-isomorphism. Consider the quotient

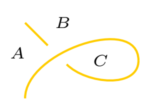
$$C/C' = \Big([\![$$  $$]\!]_{/v_+=0} \to 0 \Big)$$

The notation $[\![\ldots]\!]_{/v_+}$ denotes the quotient by all generators represented by diagrams in which the loop is labeled with a $+$. We want to construct a map $f : C/C' \to$ $[\![$  $]\!]$. On generators:

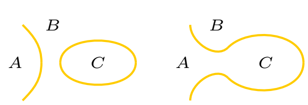


Since all generators in $[\![$  $]\!]_{/v_+=0}$ have the closed component marked with a $-$, we can see that f is an isomorphism. We can then combine this with q above to get a map $\rho_I : C = [\![$  $]\!] \to [\![$  $]\!]$, defined as $\rho_I = f \circ q$. Since q is a quasi-isomorphism and f is an isomorphism, we have shown that ρ_I is a quasi-isomorphism as well, thus completing the proof of Reidemeister I invariance.

Filtered version. We need to check that ρ_I : $[\![$  $]\!] \to [\![$  $]\!]$ is a filtered quasi-isomorphism. To help, we start by labelling components of $\Sigma \setminus D$ as below:



Extend these component labels to any other diagram obtained as a resolution of this one:



Recall that $\rho_I = f \circ q$. We will proceed to prove that each of the factors f and q are filtered quasi-isomorphisms.

The map $q : C \to C/C'$ is strict, since we defined it via a quotient by a subcomplex. Therefore, we have a short exact sequence

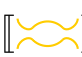
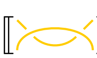
$$0 \longrightarrow \mathrm{gr}(C') \longrightarrow \mathrm{gr}(C) \xrightarrow{\mathrm{gr}(q)} \mathrm{gr}(C/C') \longrightarrow 0$$

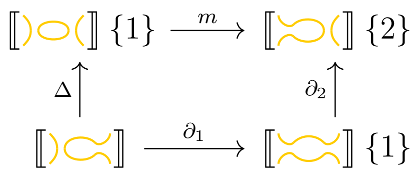
To verify that q is a filtered quasi-isomorphism, it therefore suffices to show that $\mathrm{gr}(C')$ is acyclic; the long exact sequence induced by the above would then show that $\mathrm{gr}(q)$ induces an isomorphism $H^*(\mathrm{gr}(C)) \to H^*(\mathrm{gr}(C/C'))$. Above, we showed that C' is acyclic by noticing that the differential is given by a merge map m that restricts to an isomorphism. By looking at the diagrams representing the source and target of m, we see that m leaves the grading with respect to punctures in B unchanged, and combines the gradings with respect to punctures in A and C. We only need to consider Reidemeister I moves that represent smooth isotopy in $\Sigma \times \mathbb{I}$; therefore, we can assume that there are no punctures in C, and thus that m preserves the grading of all generators. This implies that $\mathrm{gr}(m)$ is also an isomorphism, and thus that $H^*(\mathrm{gr}(C')) = 0$. This completes the proof that q is a filtered quasi-isomorphism.

We have already noted above that f is an isomorphism. From the definition of f, we see that f leaves the filtration with respect to punctures in A and B unchanged, as we assume there are no punctures in C. Therefore, f is a filtered isomorphism, which therefore implies that it is a filtered quasi-isomorphism.

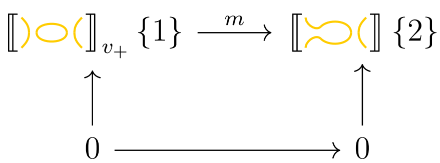
This completes the proof that ρ_I : $[\![$$]\!]$ $\to$ $[\![$$]\!]$ is a filtered quasi-isomorphism, and thus that $\mathrm{MKh}(L)$ is invariant under Reidemeister I moves.

3.7.2. Reidemeister II invariance

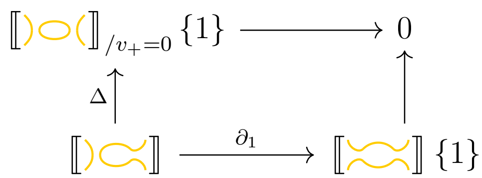
Unfiltered version. We would like to show that the complexes $[\![$$]\!]$ and $[\![$$]\!]$ have the same homology. Let $C =$ $[\![$$]\!]$ be the complex

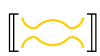


Note that ∂_1 and ∂_2 can either be merge or split maps, depending on the rest of the diagram. Let $C' \subseteq C$ be the subcomplex



Note that the merge map m in C' is an isomorphism, so C' is acyclic. Consider the quotient C/C':



We can see that $C'' =$  $\{1\}$ sits inside C/C' as a subcomplex. Additionally, the split map Δ is an isomorphism in C/C'. Consider another subcomplex $C''' \subseteq C/C'$ consisting of all $\alpha \in$  and all $(\beta, \tau\beta)$ in the graph of $\tau = \partial_1 \Delta^{-1}$:

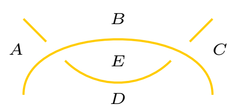
$$\begin{array}{ccc} \beta & \longrightarrow & 0 \\ \Delta\uparrow & \overset{\tau}{\searrow} & \uparrow \\ \alpha & \xrightarrow{\partial_1} & \tau\beta \end{array}$$

The map Δ in C''' is an isomorphism, so C''' is acyclic. Quotienting C/C' by C''' gives us the complex

$$\begin{array}{ccc} \beta_{/\beta=\tau\beta} & \longrightarrow & 0 \\ \uparrow & \overset{\tau}{\searrow} & \uparrow \\ 0 & \longrightarrow & \gamma \end{array}$$

Since all β are identified with some γ, an element of the above complex is uniquely determined by a choice of γ, so we get the isomorphism $(C/C')/C''' \cong C''$. Since C' and C''' are acyclic, we find that $H^*(C) \cong H^*(C'')$ as desired.

Filtered version. We need to define a map ρ_{II} :  $\to$  and check that this is a filtered quasi-isomorphism. We can define ρ_{II} as $f \circ q_2 \circ q_1$, where $q_1 : C \to C/C'$ and $q_2 : C/C' \to (C/C')/C'''$ are the two quotient maps, and $f : (C/C')/C''' \to C''$ is the isomorphism implied above. We will proceed to prove that q_1, q_2, and f are filtered quasi-isomorphisms. To help with this, label the regions of $\sigma \setminus L$ as below.



As before, extend these component labels to any other diagram obtained as a resolution of this one.

The proof that $q_1 : C \to C/C'$ is a filtered quasi-isomorphism will be very similar to the case of the map q in Section 3.7.1. Note that q_1 is strict, as it is defined via quotient by a filtered submodule. We will prove that q_1 is a filtered quasi-isomorphism by proving that $\mathrm{gr}(C')$ is acyclic. Before, we proved that C' is acyclic by noting that the differential is given by a merge m which is an isomorphism. We can now observe that m preserves the grading with respect to punctures in A, B, C, and D, and may change the grading with respect to punctures in E. Because we assume our Reidemeister II move represents smooth isotopy in $\Sigma \times \mathbb{I}$, we can assume that there are no punctures in E, and therefore that $\mathrm{gr}(m)$ is an isomorphism. Therefore, $\mathrm{gr}(C')$ is acyclic, and thus q_1 is a filtered quasi-isomorphism.

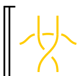
Next, we will prove that $q_2 : C/C' \to (C/C')/C'''$ is a filtered quasi-isomorphism. Again, q_2 is strict, so it suffices to prove that $\mathrm{gr}(C''')$ is acyclic. Note that the isomorphism Δ in C''' preserves the grading with respect to punctures in A, B, C, and D, and may change the grading with respect to punctures in E. By the same reasoning as above, there are no punctures in E, so Δ is a graded isomorphism.

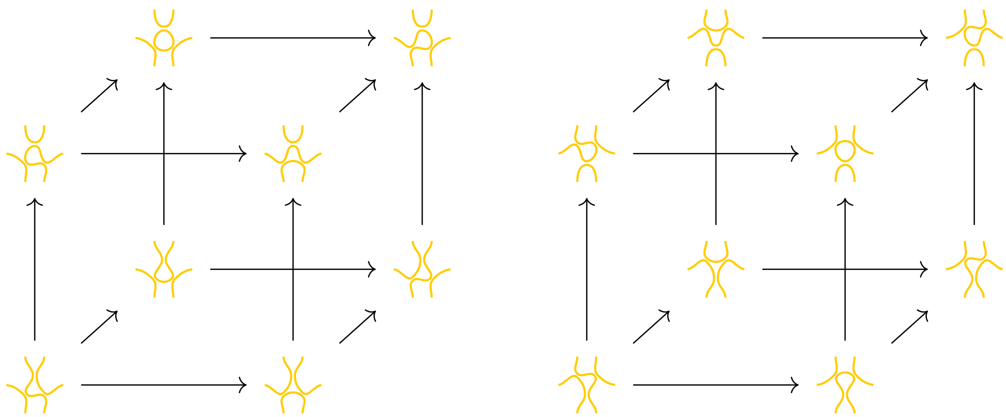
Therefore, $\mathrm{gr}(C''')$ is acyclic.

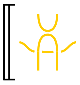
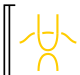
Note that $f : (C/C')/C''' \to C''$ is already an isomorphism; the only barrier to it being a filtered isomorphism is that the generators of $[\![$  $]\!]_{/v_+=0}$ are identified with certain elements of $[\![$  $]\!]$ via $\tau = \partial_1 \Delta^{-1}$. We can use Lemma A.1.3 here, and check that for any generator represented by a diagram , we cannot find a diagram  with a lower g^{Σ}-grading. Therefore, we will show that f is a filtered isomorphism by showing that τ does not increase the g^{Σ}-grading of any element. We know that the Khovanov differential ∂_1 does not increase the g^{Σ}-grading (see Lemma 3.4.1), so it remains to check that $\Delta^{-1} :$ $[\![$  $]\!]_{/v_+=0} \to [\![$  $]\!]$ does not increase the g^{Σ}-grading. We have already proven that Δ doesn't change the g^{Σ}-grading, so it follows that Δ^{-1} does not either. This means that f is a filtered isomorphism, and therefore a filtered quasi-isomorphism.

Since $\rho_{II} = f \circ q_2 \circ q_1$ and all three maps on the right-hand side are filtered quasi-isomorphisms, ρ_{II} must be as well. Therefore, our homology $\mathrm{MKh}(L)$ is invariant under Reidemeister II moves.

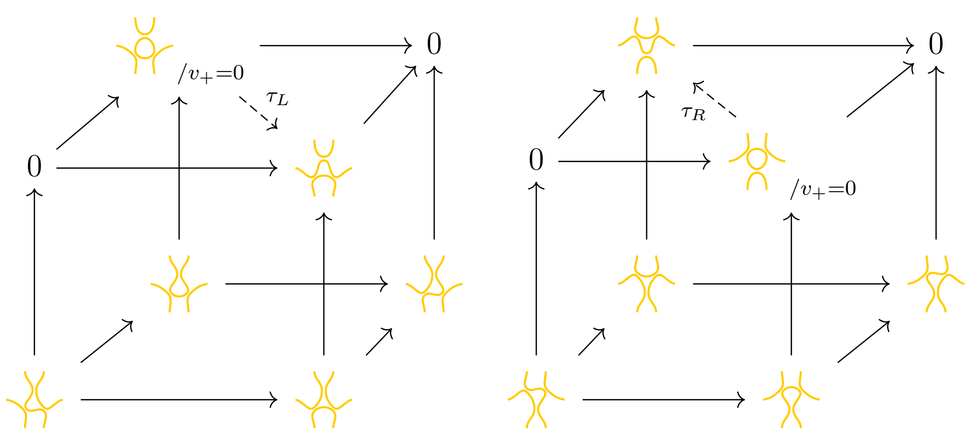
3.7.3. Reidemeister III invariance

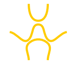
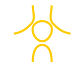
Unfiltered version. We would like to show that the complexes $L = \left[\!\!\left[\right.\right.$  $\left.\left.\right]\!\!\right]$ and $R = \left[\!\!\left[\right.\right.$  $\left.\left.\right]\!\!\right]$ have the same homology. Written out (suppressing the $[\![\,]\!]$ notation and grading shifts), these complexes are:



Note that the top level of L is isomorphic to $\left[\!\left[$  $\right]\!\right]$ and the top level of R is isomorphic to $\left[\!\left[$  $\right]\!\right]$. These diagrams are related by two Reidemeister II moves, so we can re-use some of the constructions from Section 3.7.2 here.

Specifically, if we identify the top level of L with the complex C from the previous section, the definition of $C' \subseteq C$ gives us a subcomplex $L' \subseteq L$. We can also mimic the definition of $C''' \subseteq C/C'$ to get a subcomplex $L''' \subseteq L/L'$. Note that, since L' and L''' are acyclic, $(L/L')/L'''$ has the same homology as L. We can also repeat this process on the right side by identifying the top level of R with C, and defining R' and R''' in the same way. This gives us two complexes $(L/L')/L'''$ and $(R/R')/R'''$:



Note that the bottom levels of L and R are isomorphic at each vertex since they're represented by isotopic diagrams. The two complexes above are then isomorphic via a map $\Upsilon : (L/L')/L''' \to (R/R')/R'''$ that preserves the bottom level, and transposes the top level via the isotopy  $\to$ . Note that we don't need to define Υ on elements represented by diagrams like  or  since these are already identified with other elements via the maps τ_L and τ_R. Since we have shown that $L \simeq (L/L')/L''' \cong (R/R')/R''' \simeq R$, we conclude that Kh is invariant under Reidemeister III moves.

Filtered version. Since we have already verified in Section 3.7.2 that $\mathrm{MKh}(L) \cong \mathrm{MKh}((L/L')/L''')$ and $\mathrm{MKh}(R) \cong \mathrm{MKh}((R/R')/R''')$, all we need to check is that the isomorphism Υ is a filtered isomorphism. This isn't hard to see; Υ is defined by five planar isotopies of diagrams (four on the bottom and one on the top). We can assume that there are no punctures in the center region of the diagram, and each of these isotopies can be made to avoid the punctures in the six exterior regions. Therefore, MKh is invariant under Reidemeister III moves.

Remark 3.7.3. This, along with the previous two subsections, completes the proof that $\mathrm{MKh}(L)$ is a link invariant, which makes up the first half of Theorem 3.1.1. The second half is the seemingly stronger statement that the filtered chain homotopy type of $\mathrm{CKh}(D; \Bbbk)$ is an invariant. This follows from the fact that two chain complexes of filtered projective modules are filtered quasi-isomorphic if and only if they are filtered chain homotopy equivalent (see [51, Section 03TB]). Since $\mathrm{CKh}(D)$ is a free chain complex, and the filtration $\mathcal{F}^\Sigma$ on $\mathrm{CKh}(D)$ is induced by a grading on generators, each associated graded complex $\mathrm{gr}_p^\Sigma(\mathrm{CKh}(D))$ is a chain complex of free (thus projective) $\Bbbk$-modules. Therefore, given diagrams D, D', the existence of a filtered quasi-isomorphism $\mathrm{CKh}(D \subset \Sigma) \to \mathrm{CKh}(D' \subset \Sigma)$ actually implies that the two complexes are filtered homotopy equivalent. We have constructed explicit filtered quasi-isomorphisms ρ_I and ρ_{II} realizing Reidemeister I and II moves on $\mathrm{CKh}(D)$ in Section 3.7.1 and Section 3.7.2, respectively. Our proof of invariance for Reidemeister

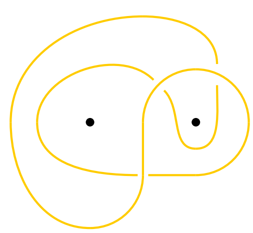
III moves does not produce a single quasi-isomorphism of free chain complexes, but does produce a zig-zag of such quasi-isomorphisms, which can be composed after realizing them as homotopy equivalences. Therefore, we have proved both halves of Theorem 3.1.1.

Section 3.8

Further examples

Since MKh(L) is in some sense constructed as a generalization of annular Khovanov homology, it is natural to ask if it is actually entirely *determined* by AKh(L) with respect to each individual puncture. This, however, is not the case; there exist non-trivial knots in punctured disks which become trivial unknots when any puncture is removed, and MKh(L) distinguishes some of these knots from the unknot. A natural class of such knots is studied in [13]. We will work out MKh(L) for one such knot here.

Example 3.8.1. Let L be the link in the twice-punctured disk depicted below.



We calculate MKh(L) to be the following:

$$\mathrm{MKh}^0(L) \cong \mathbb{Q}_{(-2,0)}\{-1\} \oplus \mathbb{Q}^2_{(0,0)}\{1\} \oplus \mathbb{Q}_{(0,-2)}\{-1\}$$
$$\mathrm{MKh}^1(L) \cong \mathbb{Q}_{(0,-2)}\{1\} \oplus \mathbb{Q}_{(-2,0)}\{1\} \oplus \mathbb{Q}_{(-2,-2)}\{-1\}$$
$$\mathrm{MKh}^2(L) \cong \mathbb{Q}_{(-2,-2)}\{1\}$$

Chapter 4

The Dowlin Spectral Sequence

Section 4.1

Introduction

In [14], Dowlin associates a filtered chain complex to a link L. The spectral sequence this filtered complex gives rise to has E_2-page isomorphic to the (reduced) Khovanov homology $\overline{\mathrm{Kh}}(L)$ and converges to the knot Floer homology $\widehat{\mathrm{HFK}}(m(L))$ of the mirror of the link. This property, that the E_2- and E_∞-pages of the spectral sequence are link invariants which do not depend on the diagram used to construct the filtered complex, suggests that maybe the same is true of all the higher pages of the spectral sequence. This is the main result of this chapter.

Theorem 4.1.1. *For $k \geq 2$, the E_k-page of Dowlin's spectral sequence does not depend on the diagram used to construct the filtered complex, and is thus a link invariant.*

This provides a family of link invariants $\{E_k(L)\}_{k=2}^{\infty}$. One future direction for this work, which is not addressed in this chapter, is to consider implications in the study of transverse links. In [42], Plamenevskaya identifies an invariant of transverse links $\psi(L) \in \mathrm{Kh}(L)$. By studying the effects of our invariance maps on this class in

Khovanov homology, one could hope to define a countable family of transverse link invariants by taking the image of ψ under Dowlin's spectral sequence, in the style of [5]. It would be interesting to compare these invariants, and notably the image of ψ on the E_∞-page $\widehat{\mathrm{HFK}}(m(L))$, with known transverse link invariants [3]. Another, perhaps related, application of these results could be to the concordance invariants s and τ. In general, we believe that proving the invariance of the higher pages of the Dowlin spectral sequence is an important step in deciphering the connection between Khovanov homology and knot Floer homology.

Organization

In Section 4.2 we review the construction of the filtered complex C_2^- inducing the spectral sequence from $\overline{\mathrm{Kh}}(L)$ to $\widehat{\mathrm{HFK}}(m(L))$ for a link L, as originally defined by Dowlin in [14]. In Section 4.3, we prove that the homotopy type of this complex is invariant under a diagrammatic change called "relabeling vertices". In Section 4.4, we define filtered chain maps between filtered complexes associated to diagrams separated by MOY moves, after recalling what these are. With these in hand, we prove invariance of the higher pages of the spectral sequence in Section 4.5.

Conventions

There are a few homological algebra conventions that we need to establish.

- We will call our complexes chain complexes, despite the fact that our differentials will usually have degree 1 with respect to the homological grading.
- Our filtrations will be *descending*, which is to say that $\mathcal{F}_i M \supseteq \mathcal{F}_j M$ when $i \leq j$.
- A *filtered quasi-isomorphism* $f : A \to B$ is a filtered chain map which induces a quasi-isomorphism between the associated graded complexes $\mathrm{gr}(f)$:

$\mathrm{gr}(A) \to \mathrm{gr}(B)$. In other words, a filtered quasi-isomorphism induces a quasi-isomorphism between E_0-pages of spectral sequences, and equivalently induces isomorphisms between E_1-pages. If A and B are connected by a zig-zag of filtered quasi-isomorphisms, then they have the same weak filtered homotopy type, a relationship which we denote by $A \simeq B$.

- Because the E_1-page of the filtered complex C_2^- is isomorphic to the Khovanov *complex*, and not the Khovanov *homology*, we will need to work with invariance maps which are not filtered quasi-isomorphisms. Instead, they only induce quasi-isomorphisms on the E_1-pages, or equivalently induce isomorphisms on the E_2-pages. We will call these maps E_1*-quasi-isomorphisms* (terminology from [12]). As above, we write $A \simeq_1 B$ to denote that A and B are connected by a zig-zag of E_1-quasi-isomorphisms.

- Since we will be working with two different notions of weak equivalence, we also need two different mapping cones for a filtered map $f : A \to B$, denoted $\mathrm{cone}(f)$ and $\mathrm{cone}_1(f)$. Both of them will have the same underlying unfiltered complex, but will differ in the definition of the filtration. The former filtration is defined to be $\mathcal{F}_i(\mathrm{cone}(f)) = \mathcal{F}_i A \oplus \mathcal{F}_i B$, whereas the latter filtration is given by $\mathcal{F}_i(\mathrm{cone}(f)) = \mathcal{F}_i A \oplus \mathcal{F}_{i-1} B$.

Acknowledgements

The authors thank Ina Petkova for suggesting this project, as well as providing helpful comments throughout, and thank John Baldwin and Nathan Dowlin for enlightening conversations.

Section 4.2

The Spectral Sequence

In this section, we review the construction of the spectral sequence from $\overline{\mathrm{Kh}}(L)$ to $\widehat{\mathrm{HFK}}(L)$ for a link L, as originally defined by Dowlin in [14]. The spectral sequence arises from a filtered chain complex $\mathrm{C}_2^-(D)$ constructed from a *partially singular braid diagram* D associated to an unoriented link L. In Section 4.2.1, we define these diagrams, and in Section 4.2.2 we associate a filtered chain complex to each such diagram. Finally, in Section 4.2.3, we discuss how to associate a partially singular braid diagram to an unoriented link L, and we characterize the set of moves connecting any two such partially singular braid diagrams.

4.2.1. Partially singular braid diagrams

In this section we will define the types of diagrams we will need to construct the spectral sequence. We start by establishing some conventions regarding braid diagrams. If D is a closed braid diagram, we can consider it as a 4-valent graph embedded in $\mathbb{R}^2$ with vertices $V(D)$ the set of crossings, and edges $E(D)$ the set of arcs connecting these crossings. This agrees with the usual way of representing link diagrams as graphs. Given a graph G, recall that a *subdivision* H of G is a graph obtained by adding 2-valent vertices along edges of G.

Definition 4.2.1. A *(closed) partially singular braid diagram* is an oriented graph embedded in $\mathbb{R}^2$ which can be obtained as a subdivision of a closed braid diagram, equipped with the following extra information:

- a labeling of every 4-valent vertex as "positive", "negative", or "singular",
- a further labeling of every singular vertex as either "fixed" or "free", and
- exactly one distinguished edge, which is called the "decorated" edge.

An *open* partially singular braid diagram is defined identically to a closed one, except that it also has $2n$ 1-valent vertices (assuming n strands) corresponding to the endpoints of the strands. When drawing partially singular braid diagrams, we indicate fixed singular vertices by drawing a circle around them, as in Figure 4.2.1; 2-valent vertices are drawn simply as dots on the strands, and the decorated edge is denoted by two small lines, as in Figure 4.2.2.

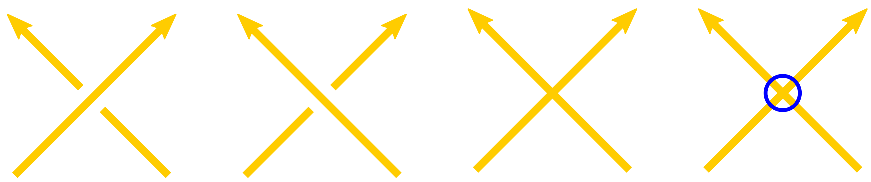


Figure 4.2.1: The different types of vertices in a partially singular braid diagram.

Figure 4.2.2: Other features that can occur in a braid diagram.

Throughout, we assume the decorated edge is leftmost in the diagram. We also assume a fixed ordering of the vertices whenever we consider a partially singular braid diagram D. We let $\text{Fixed}(D)$ denote the set of fixed singular vertices of D and $\text{Free}(D)$ denote the set of free singular vertices of D.

A *(fully) singular braid diagram* is a partially singular braid diagram with no crossings. It may arise from resolving a partially singular braid diagram D, in the following sense. Let D be a partially singular braid diagram, with $c(D)$ the set of crossings of D. Then a *resolution*, a function $I : c(D) \to \{0, 1\}$, gives a fully singular braid diagram D_I by resolving each crossing according to Figure 4.2.3. In words,

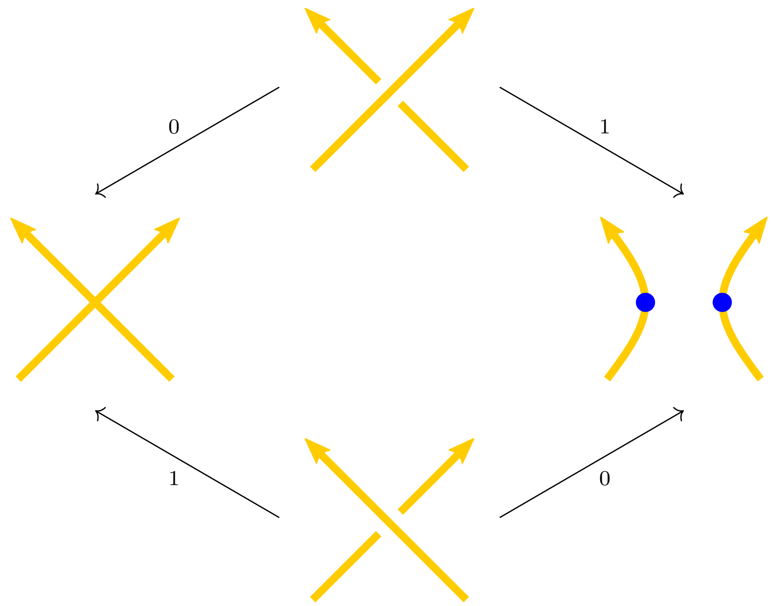


Figure 4.2.3: The 0- and 1-resolutions of positive and negative crossings.

the 0-resolution of a positive crossing is a singular vertex, and the 1-resolution is the oriented smoothing with two subdivided edges. The 0- and 1-resolutions of a negative crossing are the 1- and 0-resolutions of a positive crossing, respectively. If a fully singular braid diagram S arises as a complete resolution of a partially singular braid diagram D, then $\mathrm{Fixed}(S) = \mathrm{Fixed}(D)$, and $\mathrm{Free}(S)$ contains all crossings in $\mathrm{Free}(D)$ as well as those which were singularized in the resolution.

4.2.2. The filtered complex $\mathrm{C}_2^-(D)$

In this section, we recall Dowlin's construction of the filtered chain complex $\mathrm{C}_2^-(D)$ which gives rise to the spectral sequence connecting Khovanov homology to knot Floer homology. Throughout, let D be a partially singular braid diagram and I a resolution of D giving rise to the fully singular braid diagram D_I. We will first construct $\mathrm{C}_2^-(D_I)$ for each resolution I, then combine these into a cube complex $\mathrm{C}_2^-(D)$ by adding "edge maps".

To begin, label each edge of D by a unique integer from 1 to $k = |E(D)|$, and let

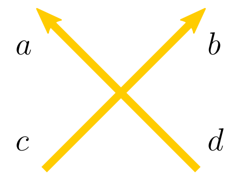


Figure 4.2.4: The local edge labels around a vertex.

$R(D) = \mathbb{Q}[U_1, \ldots, U_k]$ be the polynomial ring over $\mathbb{Q}$ generated by one variable for each edge. Note that, whenever crossings in D are resolved to get a diagram D', there is a natural bijection between edges in D and edges in D', so we can extend our edge labels to any resolution of D. To each vertex $v \in V(D)$ we associate two polynomials, $L(v)$ and $L^+(v)$. Label the adjacent edges to each vertex $v \in V(D)$ as in Figure 4.2.4; if we draw the vertex such that all edges are oriented upwards, then we label the edge in the top left by a, the remaining edges by b, c, and d as we traverse clockwise from the edge labeled a. Define $L(v) = U_a + U_b - U_c - U_d$ and $L^+(v) = U_a + U_b + U_c + U_d$.

One factor of $\mathrm{C}_2^-(D_I)$ will depend not on the specific resolution but only on D; we denote this factor $\mathcal{L}_D^+$. Let

$$\mathcal{L}_D^+ := \bigotimes_{v \in \mathrm{Fixed}(D)} R(D) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(D) \ .$$

It should be noted that $\mathcal{L}_D^+$ is not a chain complex, but rather a *matrix factorization* (or *curved complex*). A *matrix factorization* is a module M equipped with an endomorphism $\partial : M \to M$ such that $\partial^2 = \omega \, \mathrm{id}_M$ for some potentially-nonzero scalar ω, which is called the *potential* of the matrix factorization. Despite the fact that ∂ does not square to zero, we still find occasion to refer to it as a *differential* on M; this will be clear from context. In the case of $\mathcal{L}_D^+$, $\omega = \sum_{v \in \mathrm{Fixed}(D)} L(v)L^+(v)$, which is often nonzero in $R(D)$.

The factor of $\mathrm{C}_2^-(D_I)$ which is dependent on the specific resolution is the $R(D)$-

module $Q(D_I) = R(D)/(L(D_I) + N(D_I))$, where $L(D_I)$ and $N(D_I)$ are two ideals of $R(D)$. The first of these is the *linear ideal* $L(D_I)$, defined as

$$L(D_I) \coloneqq \sum_{v \in \mathrm{Free}(D_I)} (L(v)) \,.$$

The second is the *nonlocal ideal* $N(D_I)$. Let Ω be a smoothly-embedded disk in $\mathbb{R}^2$ that does not contain the decorated edge, and such that the boundary only intersects D transversely at edges. Let $\mathrm{In}(\Omega)$ (resp. $\mathrm{Out}(\Omega)$) denote the set of edges that intersect the boundary of Ω and are oriented inward (resp. outward). We define $N(\Omega)$ to be the polynomial

$$N(\Omega) \coloneqq \prod_{i \in \mathrm{Out}(\Omega)} U_i - \prod_{j \in \mathrm{In}(\Omega)} U_j \,.$$

The nonlocal ideal $N(D_I)$ is then generated by $N(\Omega)$ for all such embedded disks Ω:

$$N(D_I) \coloneqq \sum_{\Omega} (N(\Omega)) \,.$$

With the above definitions in hand, the complex $\mathrm{C}_2^-(D_I)$ is then defined as

$$\begin{aligned} \mathrm{C}_2^-(D_I) &\coloneqq Q(D_I) \otimes \mathcal{L}_D^+ \\ &\coloneqq R(D)/(L(D_I) + N(D_I)) \otimes \left(\bigotimes_{v \in \mathrm{Fixed}(D)} R(D) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(D) \right) . \end{aligned}$$

It is shown in [14, Lemma 2.4] that the potential ω of $\mathcal{L}_D^+$ is contained in $L(D_I) + N(D_I)$, and thus is zero in $Q(D_I)$. Thus, the endomorphism of $\mathrm{C}_2^-(D_I)$ induced by $\mathcal{L}_D^+$ squares to 0, so it is truly a differential; we denote it d_0.

As a module, define

$$\mathrm{C}_2^-(D) := \bigoplus_{I \in \{0,1\}^{c(D)}} \mathrm{C}_2^-(D_I)\,.$$

The differential on $\mathrm{C}_2^-(D)$ will be defined as a sum $d_0 + d_1$, where d_0 is induced by the differential d_0 on the summands $\mathrm{C}_2^-(D_I)$, and d_1 is induced by edge maps that we have yet to define. In order to do so, we must first restrict the set of partially-singular braid diagrams we are working with.

Definition 4.2.2 ([14, Definition 2.2])**.** The set $\mathcal{D}^{\mathcal{R}}$ contains all partially-singular braid diagrams D satisfying the following conditions for all $I \in \{0,1\}^{c(D)}$:

- D_I is connected, and
- the linear terms $L(v)$ for $v \in \mathrm{Free}(D_I)$ form a regular sequence[1] over $R(D)/N(D_I)$.

The latter condition is an algebraic restriction which will be used in the proof of Theorem 4.3.1. It is equivalent to the existence of an ordering $v_1, \dots, v_k$ of the vertices in $\mathrm{Free}(D_I)$ such that $L(v_j)$ is not a zero divisor in $R(D)/(N(D_I) + (L(v_1), \dots, L(v_{j-1}))$ for each $1 \leq j \leq k$. Since $R(D)$ is a graded ring and the linear terms $L(v)$ are homogeneous of positive degree, if this condition is true for one ordering of $\mathrm{Free}(D_I)$, it is true for *every* ordering.

For the rest of the definition of $\mathrm{C}_2^-(D)$, we will assume $D \in \mathcal{D}^{\mathcal{R}}$.

Let I and J be two resolutions with $I \prec J$, i.e. I and J agree on all crossings except a single $c \in c(D)$, where $I(c) = 0$ and $J(c) = 1$. Let v be the vertex corresponding to c, and label the edges adjacent to v according to Figure 4.2.4.

The edge map $d_{I,J}$ depends on whether c is a positive or negative crossing. If I and J differ at a positive crossing, let $\phi_+ : Q(D_I) \to Q(D_J)$ be the unique $R(D)$-module map such that $\phi_+(1) = 1$, and define the edge map $d_{I,J} : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ to be $d_{I,J} = \phi_+ \otimes \mathrm{id}_{\mathcal{L}_D^+}$. Else, I and J differ at a negative crossing v. In this case, let

[1] The $\mathcal{R}$ in $\mathcal{D}^{\mathcal{R}}$ likely stands for "regular".

$\phi_- : Q(D_I) \to Q(D_J)$ be the unique $R(D)$-module map such that $\phi_-(1) = U_b - U_c$, and define the edge map $d_{I,J} : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ to be $d_{I,J} = \phi_- \otimes \mathrm{id}_{\mathcal{L}_D^+}$. We may occasionally overload notation by referring to the edge map $d_{I,J}$ as $\phi_\pm$ when there is no risk of confusion.

Combine all of these maps together into a single map $d_1 : \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D)$, given by

$$d_1 := \sum_{I \prec J} \epsilon(I,J) d_{I,J} \,.$$

Here, $\epsilon(I,J)$ is a *sign assignment*, which is a labeling of the edges of the cube of resolutions by $\{\pm 1\}$ satisfying the property that every square face has an odd number of -1-labeled edges. Such a sign assignment ensures that $(d_1)^2 = 0$, and any two choices of ϵ result in isomorphic complexes. As one example, we may let $\epsilon(I,J) = (-1)^k$, where k is the number of 1's that come before the place at which I and J differ, as in [10].

Consider $\mathrm{C}_2^-(D)$ as a chain complex with total differential $d_0 + d_1$. We filter $\mathrm{C}_2^-(D)$ by weight in the cube of resolutions, i.e. the filtration on $\mathrm{C}_2^-(D)$ is given by

$$\mathcal{F}_p \mathrm{C}_2^-(D) := \bigoplus_{w(I) \geq p} \mathrm{C}_2^-(D_I) \,,$$

where $w(I) = \sum_{c \in c(D)} I(c)$ is the *weight* of I, i.e. the number of 1-resolved crossings of D_I. Note that d_0 preserves the weight, and d_1 increases it by 1, so the differential on $\mathrm{C}_2^-(D)$ is indeed filtered with respect to this decomposition.

Remark 4.2.3. We could have alternately defined $\mathrm{C}_2^-(D)$ by first defining $\mathrm{C}_2^-(S)$ for fully singular braid diagrams S, then defining $\mathrm{C}_2^-(D)$ to be the mapping cone

$$\mathrm{C}_2^-(D) := \mathrm{cone}_1(\phi \otimes \mathcal{L}_D^+) = \left(\mathrm{C}_2^-(D_0) \to \mathrm{C}_2^-(D_1)\right) \,,$$

where D_0 and D_1 above are the 0- and 1-resolutions of a particular crossing, and $\phi : Q(D_0) \to Q(D_1)$ is the associated map of quotient modules. Iterating this construction produces a filtered complex that is isomorphic to the one that we defined previously.

4.2.3. Diagrams associated to a link

Each partially singular braid diagram gives rise to an unoriented link by taking the *unoriented smoothing.*

Definition 4.2.4. Let D be a partially singular braid diagram. The *unoriented smoothing* $\mathrm{sm}(D)$ is the unoriented link obtained from D by smoothing each singular vertex in the way that does not respect the orientation.

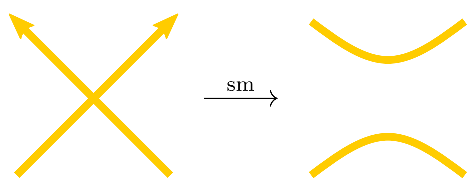


Figure 4.2.5: Unoriented smoothing of a crossing.

Figure 4.2.5 shows a local picture of smoothing a singular vertex, and Figure 4.2.6 gives an example of a partially singular braid diagram and the link obtained by taking the unoriented smoothing.

When $\mathrm{sm}(D)$ is an ℓ-component link, we can construct a "reduced" version of $\mathrm{C}_2^-(D)$. First, choose a set of edges $e_1, \ldots, e_\ell \in E(D)$ such that each e_i is on a distinct component of $\mathrm{sm}(D)$. Then, let

$$\widehat{C_2}(D) := \mathrm{C}_2^-(D) \otimes \bigotimes_{e_i} \left(R(D) \xrightarrow{U_{e_i}} R(D) \right).$$

We define the differentials given by multiplication by U_{e_i} to have weight filtration degree 1. Therefore, we get a weight filtration on $\widehat{C_2}(D)$ induced by the above definition

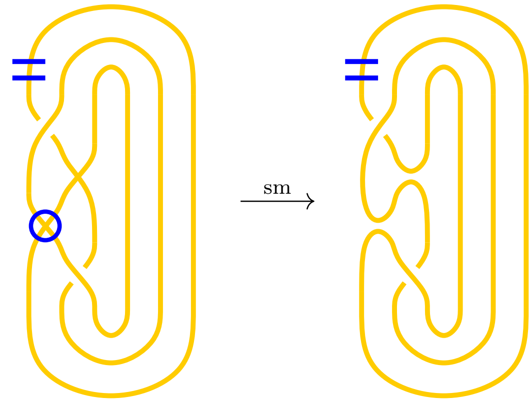


Figure 4.2.6: A diagram D and its unoriented smoothing $\mathrm{sm}(D)$.

as a tensor product of filtered complexes. This is the filtered complex that is used to define the spectral sequence relating Khovanov homology and knot Floer homology.

Theorem 4.2.5 ([14, Theorem 1.6])**.** *Let $D \in \mathcal{D}^{\mathcal{R}}$ be a partially singular braid diagram with $\mathrm{sm}(D) = L$. The spectral sequence induced by the weight filtration on $\widehat{C_2}(D)$ has E_2-page isomorphic to $\overline{\mathrm{Kh}}(L)$ and converges to $\widehat{\mathrm{HFK}}(L)$.*

In [14], Dowlin proves that every link can be realized as the unoriented smoothing of a diagram in $\mathcal{D}^{\mathcal{R}}$ by first considering a braid whose plat closure is the desired link, then turning that braid into a partially-singular braid diagram. We will go about things similarly, but instead choose a different way of embedding braid closures into $\mathcal{D}^{\mathcal{R}}$ that better fits our particular invariance proofs.

Proposition 4.2.6. *Let L be an unoriented link. There is a partially singular braid diagram $D \in \mathcal{D}^{\mathcal{R}}$ such that $\mathrm{sm}(D) = L$.*

To prove Proposition 4.2.6, we will make use of a special partially singular open braid diagram which we denote I_n. This open diagram I_n consists of $2n$ upward oriented strands with $2n-1$ layers of singular vertices. The layers are symmetric, meaning layer i has singular vertices between the same strands as layer $2n-i$ for $1 \le i < n$. The first layer has a singular vertex between the strands n and $n+1$.

The second layer has two singular vertices; one between strands $n-1$ and n and one between strands $n+1$ and $n+2$. In general, the i^{th} layer has i consecutive singular vertices, beginning with one between strands $n+1-i$ and $n+2-i$ and ending with one between strands $n-1+i$ and $n+i$. We let $\text{Fixed}(I_n)$ be the singular vertices in layers n and $n+1$, and let $\text{Free}(I_n)$ be the rest of the singular vertices. See Figure 4.2.7 for I_n in the case $n = 3$.

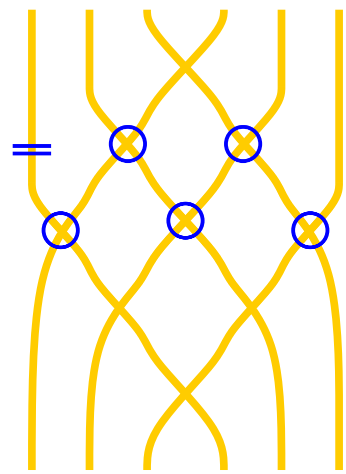


Figure 4.2.7: The partially singular braid diagram I_n in the case $n = 3$.

Definition 4.2.7. Given a braid $\beta \in B_n$, let $I_n(\beta)$ denote the partially-singular braid diagram D built by putting n downward-oriented strands to the right of β, and putting I_n above and taking the braid closure.

Proof of Proposition 4.2.6. Given an unoriented link L, let β be a braid with braid closure $\text{cl}(\beta)$ isotopic to L, the existence of which is guaranteed by Alexander's Theorem [1]. The unoriented smoothing $\text{sm}(I_n(\beta))$ is isotopic to the braid closure $\text{cl}(\beta)$ of β itself, so $D = I_n(\beta)$ is a partially singular braid diagram with $\text{sm}(D)$ isotopic to L. That $D \in \mathcal{D}^{\mathcal{R}}$ is an application of [14, Lemma 7.1]. More specifically, D contains a vertically-mirrored copy of the open braid diagram S_{2n} defined in [14], where it is proven that any such diagram is in $\mathcal{D}^{\mathcal{R}}$. □

See Figure 4.2.8 for an example of the process of constructing a partially singular braid diagram with specified unoriented smoothing.

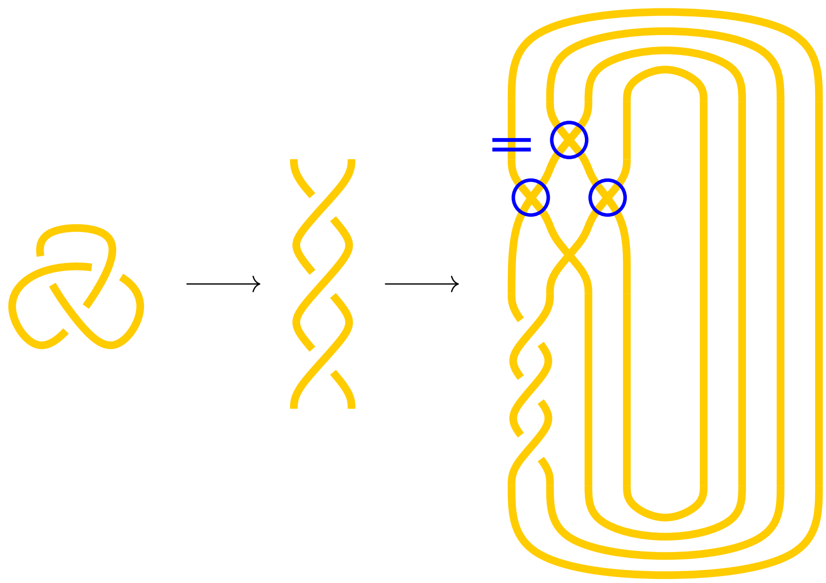


Figure 4.2.8: The process of constructing a partially singular braid diagram with smoothing isotopic to a given knot.

Let $\mathcal{D}^{\mathcal{B}} = \{I_n(\beta) \mid \beta \in B_n, n \in \mathbb{Z}\}$ be the set of partially singular braid diagrams constructed as above[2]. Then we have the following classification theorem.

Theorem 4.2.8. *Two diagrams in $\mathcal{D}^{\mathcal{B}}$ have the same unoriented smoothing if and only if the underlying braids are connected by a finite sequence of Reidemeister II moves, Reidemeister III moves, (de)stabilizations, and conjugations.*

Proof. This is just Markov's theorem, repackaged [32]. □

Since $\mathcal{D}^{\mathcal{B}} \subset \mathcal{D}^{\mathcal{R}}$, we can construct the complex $\mathrm{C}_2^-(D)$ for any $D \in \mathcal{D}^{\mathcal{B}}$. We overload notation by writing $\mathrm{C}_2^-(\beta)$ instead of $\mathrm{C}_2^-(I_n(\beta))$ for $\beta \in B_n$. We prove invariance of $\mathrm{C}_2^-(\beta)$ under the moves in Theorem 4.2.8 in Section 4.5 using maps defined in Section 4.4.

[2]Here, the $\mathcal{B}$ in $\mathcal{D}^{\mathcal{B}}$ stands for "braid".

Section 4.3

Vertex Relabeling

Before we continue towards a proof of invariance, we detour to comment on a quirk of the construction of $\mathrm{C}_2^-(D)$. One natural question to ask is why $\mathrm{C}_2^-(D)$ treats fixed and free singular vertices differently. It turns out that, in order for $H_*(\mathrm{C}_2^-(D))$ to be isomorphic to $\widehat{\mathrm{HFK}}(\mathrm{sm}(D))$, our diagram D needs to be in $\mathcal{D}^{\mathcal{R}}$, which means satisfying the regular sequence condition. This condition cannot be satisfied unless D contains sufficiently many fixed vertices in a sufficiently nice arrangement. On the other hand, we only know how to define the edge maps $d_{I,J}$ on free vertices, so we cannot make all of our vertices fixed either.

As a sort of compromise, we choose some of our vertices to be fixed and some to be free. We will not need to worry about which choice we have made when proving invariance under Reidemeister moves II and III in Section 4.5, since they only involve local pictures of diagrams which contain some crossings but no singular vertices. While not a local move, we define stabilization to be compatible with our vertex labeling as well. Conjugation, however, will require us to change which vertices are fixed and which are free; this is what motivates the following theorem.

While it is not immediately obvious, it turns out that the homotopy type of $\mathrm{C}_2^-(D)$ does not depend on the particular labeling of vertices as fixed or free in the following sense:

Theorem 4.3.1. *If $D, D' \in \mathcal{D}^{\mathcal{R}}$ are identical partially singular braid diagrams up to relabeling of fixed and free vertices, then $C_2^-(D) \simeq C_2^-(D')$.*

To prove this, we need to introduce a slight variation on the technique of "excluding a variable" from [48, Lemma 3.8] or [24, Proposition 9]. Both sources are also good references for the relevant details on matrix factorizations, including the statement below on the effect of change of basis on matrix factorizations.

We include the necessary details on matrix factorizations below. Let R be a ring. For $a, b \in R$, let $\{a, b\}$ denote the matrix factorization

$$\{a,b\} := R \underset{a}{\overset{b}{\rightleftarrows}} R .$$

For $\vec{a}, \vec{b} \in R^n$, let $\{\vec{a}, \vec{b}\} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$ denote the matrix factorization

$$\{\vec{a}, \vec{b}\} := \bigotimes_{i=1}^{n} \{a_i, b_i\} = \bigotimes_{i=1}^{n} R \underset{a_i}{\overset{b_i}{\rightleftarrows}} R .$$

We have already seen a matrix factorization of this form; if we let $\vec{a} = (L^+(v_1), \ldots, L^+(v_n))$ and $\vec{b} = (L(v_1), \ldots, L(v_n))$ for a partially singular braid diagram D with $\text{Fixed}(D) = \{v_1, \ldots, v_n\}$, then $\mathcal{L}_D^+ = \{\vec{a}, \vec{b}\}$. By definition, the potential ω associated to the matrix factorization $\{\vec{a}, \vec{b}\}$ is $\vec{a} \cdot \vec{b} = a_1 b_1 + \ldots + a_n b_n$.

A change of basis for R^n gives us an equivalent matrix factorization. One can check what effect various change of basis operations have on the representing matrix $\{\vec{a}, \vec{b}\}$. Below, we will need just one change-of-basis operation: sending $\vec{e_i}$ to $\vec{e_i} + c\vec{e_j}$ for standard basis vectors $\vec{e_i}$ and $\vec{e_j}$ of R^n. This has the effect of replacing the matrix factorization by $\{\vec{a}', \vec{b}'\}$, where

$$\vec{a}'_k = \begin{cases} \vec{a}_k + c\vec{a}_j & \text{if } k = i \\ a_k & \text{else} \end{cases}$$

and

$$\vec{b}'_k = \begin{cases} \vec{b}_k - c\vec{b}_i & \text{if } k = j \\ b_k & \text{else} \end{cases}.$$

For more details, see [48, 24].

Let $C = \{\vec{a}, \vec{b}\}$ be any matrix factorization over R. We can decompose

$$C = C' \underset{a_1}{\overset{b_1}{\rightleftarrows}} C' \ ,$$

where $C' = \{\vec{a}', \vec{b}'\}$ is the factorization obtained by omitting the first components of $\vec{a}$ and $\vec{b}$. Define $\pi : C \to C' \otimes R/(b_1)$ by $\pi((c_1, c_2)) = c_2 \otimes 1$. Before proving Theorem 4.3.1, we prove that if the potential of C is 0 and b_1 is a non-zero-divisor in R, then π is a quasi-isomorphism.

Lemma 4.3.2. *If the potential of C is 0 and b_1 is a non-zero-divisor in R, then π is a quasi-isomorphism of chain complexes.*

Proof. It is clear that π is surjective; since b_1 is a non-zero-divisor, multiplication by b_1 is injective, so we have the following short exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C' & \xrightarrow{1} & C' & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow\downarrow & & {\scriptstyle a_1 b_1}\uparrow\downarrow{\scriptstyle 1} & & {\scriptstyle a_1}\uparrow\downarrow{\scriptstyle b_1} & & \uparrow\downarrow & & \uparrow\downarrow \\
0 & \longrightarrow & C' & \xrightarrow{b_1} & C' & \longrightarrow & C' \otimes R/(b_1) & \longrightarrow & 0\,.
\end{array}$$

Let C'' denote the first nonzero column in this sequence, the matrix factorization $C' \underset{a_1 b_1}{\overset{1}{\rightleftarrows}} C'$. By the corresponding long exact sequence in homology, it suffices to show that C'' is acyclic in order to prove that π is a quasi-isomorphism. We write C''

in matrix form, then apply our above remarks about change of basis:

$$\begin{pmatrix} a_1b_1 & 1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \sim \begin{pmatrix} a_1b_1 + a_2b_2 & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \sim \begin{pmatrix} \omega & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & 0 \end{pmatrix}.$$

Since we know that the potential $\omega = 0$, we see that $C'' = \{\vec{a}', \vec{0}\} \underset{0}{\overset{1}{\rightleftarrows}} \{\vec{a}', \vec{0}\}$, and therefore is acyclic. □

With this lemma, we can now prove that $C_2^-(D)$ is independent of vertex labeling for $D \in \mathcal{D}^{\mathcal{R}}$:

Proof of Theorem 4.3.1. Let $S \in \mathcal{D}^{\mathcal{R}}$ be a fully singular braid diagram, and let $w \in \text{Fixed}(S)$ be some fixed vertex such that if w were instead free, the new diagram S' would still be in $\mathcal{D}^{\mathcal{R}}$. Note that $R(S') = R(S)$, and $Q(S') = Q(S)/(L(w))$. Since $C_2^-(S) = Q(S) \otimes \mathcal{L}_S^+$, we may consider $C_2^-(S)$ as the matrix factorization $\{\vec{a}, \vec{b}\}$ over $R = Q(S)$ with $\vec{a} = (L^+(v))_{v \in \text{Fixed}(S)}$ and $\vec{b} = (L(v))_{v \in \text{Fixed}(S)}$. Assume without loss of generality that $b_1 = L(w)$. Since $S' \in \mathcal{D}^{\mathcal{R}}$, we know that the linear terms $L(v)$ for $v \in \text{Free}(S')$ form a regular sequence over $R(S')/N(S') = R(S)/N(S)$, and in particular, $L(w)$ is a non-zero-divisor in $Q(S)$, since $w \in \text{Free}(S')$. We then get that

$$\begin{aligned}
\mathrm{C}_2^-(S) &= Q(S) \otimes_{R(S)} \mathcal{L}_S^+ \\
&\cong \{\vec{a}, \vec{b}\} \\
&\simeq Q(S)/(L(w)) \otimes_{Q(S)} \Big(Q(S) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} \Big) && \text{(by Lemma 4.3.2)} \\
&\simeq \big(Q(S)/(L(w)) \otimes_{Q(S)} Q(S) \big) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} && \text{(by associativity of } \otimes) \\
&\simeq Q(S)/(L(w)) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} \\
&\cong Q(S') \otimes_{R(S)} \mathcal{L}_{S'}^+ \\
&\cong Q(S') \otimes_{R(S')} \mathcal{L}_{S'}^+ && \text{(since } R(S) = R(S')) \\
&= \mathrm{C}_2^-(S') \,.
\end{aligned}$$

Therefore, we see that changing a fixed vertex to a free one in a fully singular diagram does not change the homotopy type of $\mathrm{C}_2^-(-)$ as long as both diagrams are in $\mathcal{D}^{\mathcal{R}}$.

Now, we need to extend this result. Let $D, D' \in \mathcal{D}^{\mathcal{R}}$ be partially singular braid diagrams that differ only on the labeling of a single vertex $w \in \text{Fixed}(D) \cap \text{Free}(D')$. We know that $\mathrm{C}_2^-(D_I) \simeq \mathrm{C}_2^-(D'_I)$ for all $I \in \{0,1\}^{c(D)}$. In particular, we have a map in one direction: $\pi : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ is a filtered quasi-isomorphism inducing the above equivalence. Therefore, it suffices to show that π commutes with the edge map d_1, which is the sum of $d_{I,J}$. Since π is linear over $Q(S)$, we get that it is additionally $R(S)$-linear via the natural quotient map, and therefore commutes with scalar multiplication by elements of $R(S)$. Since the edge maps $d_{I,J}$ are defined via scalar multiplication by 1 or $U_b - U_c$, we see that π does in fact commute with the edge maps, and therefore extends to a filtered quasi-isomorphism $\pi : \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ by Lemma A.2.4.

Given any two diagrams $D', D'' \in \mathcal{D}^{\mathcal{R}}$ that differ only by some number of vertex labels, we can construct a diagram $D \in \mathcal{D}^{\mathcal{R}}$ with $\text{Fixed}(D) = \text{Fixed}(D') \cup \text{Fixed}(D'')$, and therefore get that $\mathrm{C}_2^-(D') \simeq \mathrm{C}_2^-(D) \simeq \mathrm{C}_2^-(D'')$, thus proving the general case. □

Section 4.4

MOY Moves

In [35], Murakami, Ohtsuki, and Yamada study local operations on singular diagrams ("MOY moves"). While originally formulated for oriented planar trivalent graphs, they are relevant to us because one can think of singular vertices in our braids and braid resolutions as pairs of trivalent vertices instead. Two of these moves, MOY I and MOY III, represent planar isotopy when applied to the unoriented smoothing of a diagram, and thus are useful to make up for the fact that we cannot isotope singularized crossings in the same ways that we can smoothed ones. The MOY II move corresponds to a cup/cap cobordism, but is rather more limited in its application. Nevertheless, these three moves will suffice to define Reidemeister moves (and others) in Section 4.5. The maps that we choose to realize these moves are inspired by those used in [24] and [26].

In this section, we construct filtered chain maps relating $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$, where D and D' are partially singular braid diagrams connected by an MOY I, II, or III move.

4.4.1. MOY I

Suppose D and D' are partially singular braid diagrams that differ by an MOY I move, as illustrated in Figure 4.4.1. In words, there is a fixed vertex v in D that meets the same edge e twice; without loss of generality, e is to the right of v. The diagram D' is then obtained from D by removing the edge e and relabeling v as a bivalent vertex.

Theorem 4.4.1. *There exist $R(D')$-linear filtered quasi-isomorphisms $\mu_{\mathrm{I}} : C_2^-(D) \to$*

$C_2^-(D')$ and $\mu_{\mathrm{I}}' : C_2^-(D') \to C_2^-(D)$. Under the identification $E_1(C_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, these maps induce the expected isomorphisms corresponding to planar isotopy.

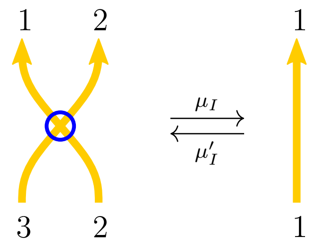


Figure 4.4.1: An MOY I move.

First, suppose S and S' are fully singular braid diagrams that differ by an MOY I move, as illustrated in Figure 4.4.1. Specifically, S contains a fixed singular vertex v that meets the same edge twice. We would like to construct filtered chain maps $\mu_{\mathrm{I}} : \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ and $\mu_{\mathrm{I}}' : \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$. To start, let us characterize $\mathrm{C}_2^-(S)$ and $\mathrm{C}_2^-(S')$.

Without loss of generality, assume that the edge which is deleted by the MOY I move is to the right of the vertex. Label this edge with the variable U_2, label the top left edge U_1, and label the bottom left edge with U_3, again as in Figure 4.4.1.

Let R be the polynomial ring over all edges not shown in the local diagram; thus, $R(S') = R[U_1]$ and $R(S) = R(S')[U_2, U_3]$. We relate the associated quotient rings by the following proposition:

Proposition 4.4.2. *As $R(S')$ modules, $Q(S') \cong Q(S)/(U_1 + U_2)$.*

Proof. We expand the right-hand side as a quotient of a free $R(S')$-module:

$$\begin{aligned} Q(S)/(U_1+U_2) &\cong Q(S)\otimes_{R(S)} R(S)/(U_1+U_2) \\ &\cong R(S)/(L(S)+N(S))\otimes R(S)/(U_1+U_2) \\ &\cong R(S)/(L(S)+N(S)+(U_1+U_2)) \\ &\cong R(S')[U_2,U_3]/(L(S)+N(S)+(U_1+U_2)) \\ &\cong R(S')[U_2,U_3]/(L(S)+\tilde{N}(S)+(U_2-U_3)+(U_1+U_2)) \\ &\cong R(S')/(L(S)+\tilde{N}(S))\,. \end{aligned}$$

In the above, $\tilde{N}(S)$ is the sum of the non-local relations other than U_1-U_3; this is exactly equal to $N(S')$, as any region intersecting these local diagrams can be made to avoid U_2 and any intersections with U_3 can be isotoped to intersect U_1 instead. Further, $L(S)=L(S')$. Thus we have $Q(S)/(U_1+U_2)\cong R(S')/(L(S)+\tilde{N}(S)) = R(S')/(L(S')+N(S'))=Q(S')$, as desired. □

Proposition 4.4.3. *The chain complexes $C_2^-(S)$ and $C_2^-(S')$ are quasi-isomorphic as complexes over $R(S')$.*

Proof. We can use Proposition 4.4.2 to expand $\mathrm{C}_2^-(S)$:

$$\begin{aligned}
\mathrm{C}_2^-(S) &= Q(S) \otimes_{R(S)} \mathcal{L}_S^+ \\
&= Q(S) \otimes \left(R(S) \underset{U_1+2U_2+U_3}{\overset{U_1-U_3}{\rightleftarrows}} R(S) \otimes \widetilde{\mathcal{L}_S^+} \right) \\
&= Q(S) \otimes \left(R(S) \underset{2U_1+2U_2}{\overset{0}{\rightleftarrows}} R(S) \otimes \widetilde{\mathcal{L}_S^+} \right) && \text{(using relation } U_1 - U_3 \text{ in } N(S)) \\
&\simeq Q(S) \otimes \left(R(S)/(U_1+U_2) \otimes \widetilde{\mathcal{L}_S^+} \right) && \text{(replacing } 2U_1+2U_2 \text{ by the cokernel)} \\
&\cong (Q(S) \otimes R(S)/(U_1+U_2)) \otimes \widetilde{\mathcal{L}_S^+} \\
&\cong Q(S') \otimes_{R(S')} \mathcal{L}_{S'}^+ && \text{(by Proposition 4.4.2)} \\
&= \mathrm{C}_2^-(S').
\end{aligned}$$

In the above, let $\widetilde{\mathcal{L}_S^+} = \bigotimes_{w \in \mathrm{Fixed}(D) \setminus \{v\}} R(D) \underset{L^+(w)}{\overset{L(w)}{\rightleftarrows}} R(D)$, and note $\widetilde{\mathcal{L}_S^+} = \mathcal{L}_{S'}^+$. Note that we may replace the mapping cone of $2U_1 + 2U_2$ by its cokernel in the fourth line only after checking that $2U_1 + 2U_2$ is not a zero divisor in $Q(S)$; by the logic in the proof of Proposition 4.4.2, we may choose a generating set of relations for $N(S) + L(S)$, none of which contain a term with a nonzero power of U_2. Therefore, $Q(S)$ is isomorphic to a free polynomial ring over U_2; since $2U_1+2U_2$ is a unit multiple (over $\mathbb{Q}$) of a monic polynomial in U_2, we therefore get that it is not a zero divisor in $Q(S)$. □

Let $\mu_{\mathrm{I}} : \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ be the quotient map implied by the above calculations. Explicitly on a simple tensor, $\mu_{\mathrm{I}}([r] \otimes (a,b) \otimes \tilde{s}) = [rb] \otimes \tilde{s}$. Let $\mu_{\mathrm{I}}' : \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ be the splitting of μ_{I} given by inclusion into the first $R(S)$ summand in the equivalence of $R(S)/(U_1 + U_2)$ and $R(S) \underset{2U_1+2U_2}{\overset{0}{\rightleftarrows}} R(S)$ in the above proof. Explicitly on a simple tensor, $\mu_{\mathrm{I}}'([r] \otimes \tilde{s}) = [r] \otimes (0,1) \otimes \tilde{s}$.

For partially singular braid diagrams D and D' related by an MOY I move, extend both maps to the cube of resolutions by defining $\mu_{\mathrm{I}} : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{I}} : \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.4.1. It is clear that μ_{I} and μ'_{I} are filtered maps, since they are defined component-wise on the cube of resolutions.

We need to check that μ_{I} and μ'_{I} are chain maps, i.e. that they commute with the edge maps d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathcal{L}_D^+ = 1 \otimes \mathcal{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathcal{L}_D^+ = (U_b - U_c) \otimes \mathcal{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{II} and μ'_{II} were defined to be $R(D')$-linear, we get that they commute with d_1. □

4.4.2. MOY II

Suppose D and D' are partially singular braid diagrams with D' the result of applying an MOY II move to D and reducing the number of crossings, as shown in Figure 4.4.2. In words, D contains a free vertex v_1, a fixed vertex v_2, and two edges e_5 and e_6 from v_2 to v_1. The diagram D' is obtained from D by removing e_5 and e_6 and merging v_1 and v_2 into a single fixed vertex.

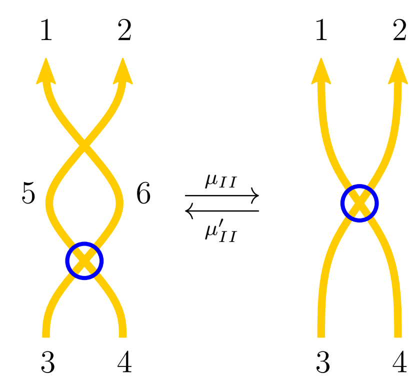


Figure 4.4.2: An MOY II move.

Theorem 4.4.4. *There exists a direct sum decomposition* $C_2^-(D) \cong C_2^-(D') \oplus C_2^-(D')$ *as filtered chain complexes over* $R(D')$*. Define* $\mu_{II} : C_2^-(D) \to C_2^-(D')$ *to be projection onto the second summand, and define* $\mu'_{II} : C_2^-(D') \to C_2^-(D)$ *to be inclusion into the first summand. Under the identification* $E_1(C_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$*, the maps* μ_{II} *and* μ'_{II} *induce the maps on Khovanov homology corresponding to the cobordisms which delete and introduce a circle, respectively.*

To start, let S and S' be fully singular braid diagrams again with S' the result of applying an MOY II move to S reducing the number of crossings. Let R be the polynomial ring over all edges not shown in the local diagrams, so that $R(S') = R[U_1, U_2, U_3, U_4]$, and $R(S) = R(S')[U_5, U_6]$.

Proposition 4.4.5. *As an* $R(S')$*-module,* $Q(S) \cong Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle$*.*

Proof. First, note that $Q(S) = Q(S')[U_5, U_6]/(U_5 + U_6 - U_1 - U_2, U_5U_6 - U_1U_2)$. We do not need to consider any other non-local relations, as any region Ω intersecting these diagrams can be isotoped away from U_5 and U_6 to give an equivalent or stronger relation. We want to prove that $\{1, U_6\}$ is a basis for $Q(S)$ over $Q(S')$. To see that $\{1, U_6\}$ is a generating set, it is enough to note that in $Q(S)$, $U_5 = U_1 + U_2 - U_6$, and that $(U_1 + U_2 - U_6)U_6 - U_1U_2 = 0$, so $U_6^2 = (U_1 + U_2)U_6 - U_3U_4$. Linear independence follows from the fact that $U_6^2 - (U_1 + U_2)U_6 + U_1U_2$ is a monic polynomial of degree 2 in U_6. □

Using this proposition, we can decompose

$$\begin{aligned}
\mathrm{C}_2^-(S) &\cong Q(S) \otimes_{R(S)} \mathcal{L}_S^+ \\
&\cong Q(S) \otimes_{R(S)} \left(\bigotimes_{v \in \mathrm{Fixed}(S)} R(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(S) \right) \\
&\cong Q(S) \otimes_{R(S)} \left(\bigotimes_{v \in \mathrm{Fixed}(S')} R(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(S) \right) \\
&\cong \bigotimes_{v \in \mathrm{Fixed}(S')} Q(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S) \\
&\cong \bigotimes_{v \in \mathrm{Fixed}(S')} Q(S')\langle 1 \rangle \oplus Q(S')\langle U_6 \rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle 1 \rangle \oplus Q(S')\langle U_6 \rangle \\
&\cong \left(\bigotimes_{v \in \mathrm{Fixed}(S')} Q(S')\langle 1 \rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle 1 \rangle \right) \oplus \left(\bigotimes_{v \in \mathrm{Fixed}(S')} Q(S')\langle U_6 \rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle U_6 \rangle \right) \\
&\cong \left(Q(S')\langle 1 \rangle \otimes_{R(S')} \mathcal{L}_{S'}^+ \right) \oplus \left(Q(S')\langle U_6 \rangle \otimes_{R(S')} \mathcal{L}_{S'}^+ \right) \\
&\cong \mathrm{C}_2^-(S')\langle 1 \rangle \oplus \mathrm{C}_2^-(S')\langle U_6 \rangle \ .
\end{aligned}$$

Define $\mu_{\mathrm{II}} : \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ to be projection onto the second summand in the above decomposition, and define $\mu'_{\mathrm{II}} : \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ to be inclusion into the first summand. For partially singular braid diagrams D and D' related by an MOY II move, extend both maps to the cube of resolutions by defining $\mu_{\mathrm{II}} : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{II}} : \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.4.4. It is clear that μ_{II} and μ'_{II} are filtered maps, since they are defined component-wise on the cube of resolutions. Next, we need to check that μ_{II} and μ'_{II} are chain maps, i.e. that they commute with the edge maps d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathcal{L}_D^+ = 1 \otimes \mathcal{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathcal{L}_D^+ = (U_b - U_c) \otimes \mathcal{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{II}

and μ'_{II} were defined to be $R(D')$-linear, we get that they commute with d_1.

We used a direct sum decomposition of $\mathrm{C}_2^-(S)$ to define these maps on complete resolutions. We can see this direct sum decomposition on the cube of resolutions as well. Specifically, we have a split exact sequence:

$$0 \longrightarrow \mathrm{C}_2^-(D') \underset{1\otimes\mathcal{L}_D^+}{\overset{\mu'_{\mathrm{II}}}{\rightleftarrows}} \mathrm{C}_2^-(D) \underset{(U_6-U_1)\otimes\mathcal{L}_D^+}{\overset{\mu_{\mathrm{II}}}{\rightleftarrows}} \mathrm{C}_2^-(D') \longrightarrow 0\ .$$

Finally, we want to show that these maps induce the correct morphisms on the Khovanov complex. The cobordism corresponding to the introduction of a circle is induced by multiplication by 1 [9]. This should correspond to μ'_{II}, which we can see also induces multiplication by 1 on homology. The cobordism corresponding to the deletion of a circle should send $1 \mapsto 0$ and $X \mapsto 1$, where X is a variable associated to the shrinking circle. In our case, μ_{II} maps $1 \mapsto 0$ and $U_6 \mapsto 1$, inducing this same map on homology. □

One can repeat the same argument to show that we also have similar MOY II decompositions for the cases in Figure 4.4.3.

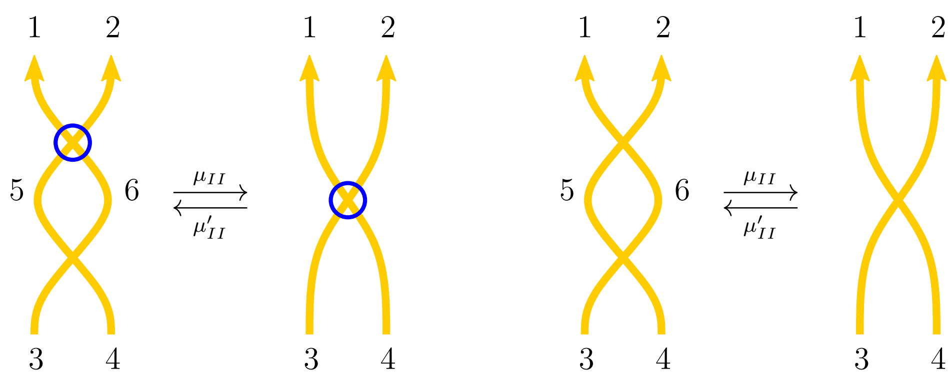


Figure 4.4.3: Variations of the MOY II move.

4.4.3. MOY III

Suppose D and D' are fully singular braid diagrams with D' the result of applying an MOY III move to D and reducing the number of crossings, as shown in Figure 4.4.4. In words, D contains a fixed vertex v_1, free vertices v_2 and v_3, and edges $e_7 : v_2 \to v_1$, $e_8 : v_3 \to v_1$, and $e_9 : v_3 \to v_2$. The diagram D' is obtained from D by removing the edges e_7, e_8, and e_9, merging v_1 and v_3 into a single fixed vertex, removing v_2, and merging e_6 into e_3.

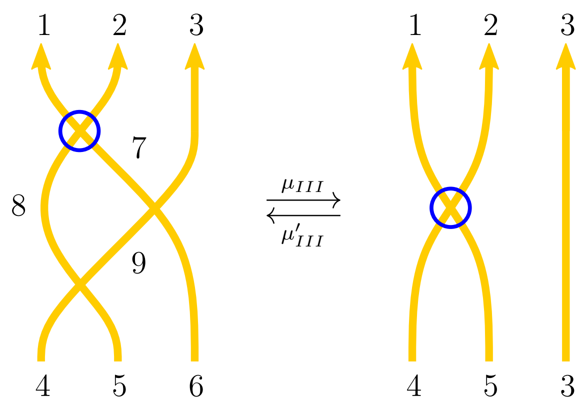


Figure 4.4.4: An MOY III move.

Theorem 4.4.6. *There exist $R(D')$-linear filtered quasi-isomorphisms $\mu_{\text{III}} : C_2^-(D) \to C_2^-(D')$ and $\mu'_{\text{III}} : C_2^-(D') \to C_2^-(D)$. Furthermore, $C_2^-(D')$ is isomorphic to a direct summand of $C_2^-(D)$. Under the identification $E_1(C_2^-(-)) \cong \text{CKh}^-(\text{sm}(-))$, these maps induce the expected isomorphisms corresponding to planar isotopy.*

As for MOY I and II, we will again start by defining these maps on fully singular braid diagrams then extending them to the cube of resolutions. Let S and S' be fully singular braid diagrams with S' the result of applying an MOY III move to S reducing the number of crossings as in Figure 4.4.4. We will prove that $\text{C}_2^-(S) \cong \text{C}_2^-(S') \oplus \Upsilon_L$, where Υ_L is some acyclic complex. Furthermore, the MOY III move has a nontrivial horizontal mirroring. We will prove that in the case where S and S' are connected by an MOY III move which is the mirror of Figure 4.4.4, we have $\mathrm{C}_2^-(S) \cong \mathrm{C}_2^-(S') \oplus \Upsilon_R$. While it is true that $\Upsilon_R = \Upsilon_L$, we will neither need this fact nor prove it in this thesis. Nevertheless, we may refer to the complex as Υ regardless.

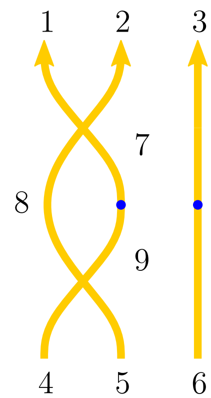


Figure 4.4.5: The fully singular diagram S'' used in the definitions of μ_{III} and μ'_{III}.

We will construct a map $\mu_{\mathrm{III}} : \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ and another map $\mu'_{\mathrm{III}} : \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ which splits μ_{III}, thus proving that $\mathrm{C}_2^-(S')$ is a direct summand of $\mathrm{C}_2^-(S)$. Let S'' be the fully singular braid diagram where the middle singular vertex v_2 is replaced by the oriented smoothing as in Figure 4.4.5, so that we may define the map $1 \otimes \mathcal{L}_S^+ : \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S'')$. We may then apply an MOY II move μ_{II} to the left two strands in S'' to get a map $\mu_{II} : \mathrm{C}_2^-(S'') \to \mathrm{C}_2^-(S')$. Therefore, we define $\mu_{\mathrm{III}} = \mu_{\mathrm{II}} \circ (1 \otimes \mathcal{L}_S^+)$. We can also reverse the order of these operations to define $\mu'_{\mathrm{III}} = ((U_9 - U_3) \otimes \mathcal{L}_D^+) \circ \mu'_{\mathrm{II}}$. Note that the maps $1 \otimes \mathcal{L}_S^+$ and $(U_9 - U_3) \otimes \mathcal{L}_S^+$ are well-defined since if v_2 were replaced by a positive or negative crossing, these would simply be multiples of the edge maps corresponding to resolutions of that crossing.

Proposition 4.4.7. μ_{III} *splits* μ'_{III}, *i.e.* $\mu_{\mathrm{III}} \circ \mu'_{\mathrm{III}} = \mathrm{id}_{C_2^-(S')}$

Proof. We expand out the definitions of μ_{III} and μ'_{III} to get

$$\begin{aligned}
\mu_{\text{III}} \circ \mu'_{\text{III}} &= \mu_{\text{II}} \circ (1 \otimes \mathcal{L}_S^+) \circ ((U_9 - U_3) \otimes \mathcal{L}_S^+) \circ \mu'_{\text{II}} \\
&= \mu_{\text{II}} \circ ((U_9 - U_3) \otimes \mathcal{L}_S^+) \circ \mu'_{\text{II}} \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -U_3 & -U_4U_5 \\ 1 & U_4 + U_5 - U_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathcal{L}_S^+ \\
&= (1) \otimes \mathcal{L}_S^+ \\
&= \mathrm{id}_{\mathrm{C}_2^-(S')} .
\end{aligned}$$

□

Therefore, we get a direct sum decomposition $\mathrm{C}_2^-(S) \cong \Upsilon \langle 1 \rangle \oplus \mathrm{C}_2^-(S') \langle U_9 - U_3 \rangle$. For partially singular braid diagrams D and D' related by an MOY III move, extend both maps to the cube of resolutions by defining $\mu_{\text{III}} : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\text{III}} : \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.4.6. It is clear that μ_{III} and μ'_{III} are filtered maps, since they are defined component-wise on the cube of resolutions. Furthermore, we can extend our proof of Proposition 4.4.7 to partially-singular braid diagrams since both the edge maps and MOY II maps are defined on such diagrams, so $\mathrm{C}_2^-(D')$ really is a summand of $\mathrm{C}_2^-(D)$.

We also need to check that μ_{III} and μ'_{III} are chain maps, i.e. that they commute with the edge maps d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathcal{L}_D^+ = 1 \otimes \mathcal{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathcal{L}_D^+ = (U_b - U_c) \otimes \mathcal{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{III} and μ'_{III} were defined to be $R(D')$-linear, we get that they commute with d_1. □

As before, a similar argument shows that we also have MOY III decompositions for the cases in Figure 4.4.6.

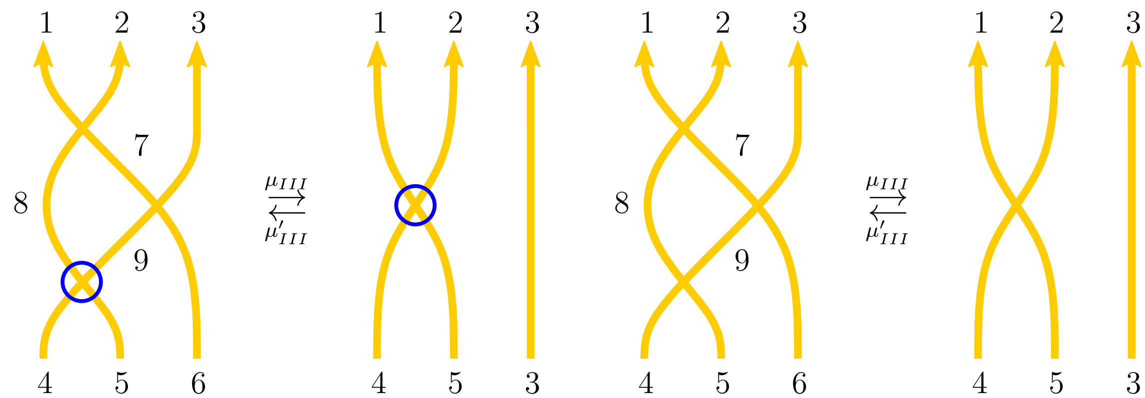


Figure 4.4.6: Variations of the MOY III move.

Section 4.5

Invariance

In this section, we prove that $C_2^-(\beta)$ is an invariant of the braid closure $\mathrm{cl}(\beta)$ by showing that it is invariant under each of the four moves of Theorem 4.2.8. The first two moves, Reidmeister II and III, apply to any partially singular braid diagram D, whereas the second two moves, stabilization and conjugation, are specific to diagrams of the form $D = I_n(\beta)$.

4.5.1. Reidemeister II

We begin by proving invariance under Reidemeister II moves. There are two distinct such moves, but they are mirror images of each other, and their proofs are almost identical. We prove one of the cases in detail below.

Theorem 4.5.1. *If D and D' are two partially singular braid diagrams that differ by a Reidemeister II move, then $C_2^-(D) \simeq_1 C_2^-(D')$ over $R(D')$.*

Proof. Let D and D' be the diagrams in Figure 4.5.1, with D' the result of eliminating two crossings from D' by means of a Reidemeister II move. We will use Lemma A.2.1

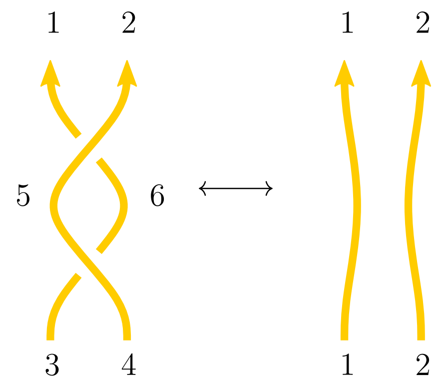


Figure 4.5.1: A Reidemeister II move.

to simplify $C_2^-(D)$ and $C_2^-(D')$ to see they have the same homotopy type. Label the edges of D with variables $U_1, \ldots, U_6$ as in Figure 4.5.1, and order the crossings from top to bottom. Let $\phi_1 = \phi_+ = 1$ be the edge map corresponding to the top (positive) vertex, and let $\phi_2 = \phi_- = U_6 - U_3$ be the edge map corresponding to the bottom (negative) vertex. Then, fixing a sign assignment without loss of generality, we expand the cube of resolutions for $C_2^-(D)$ as:

$$\begin{array}{ccc} C_2^-(D_{00}) & \xrightarrow{\phi_1} & C_2^-(D_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle -\phi_2} \\ C_2^-(D_{01}) & \xrightarrow{\phi_1} & C_2^-(D_{11}) . \end{array}$$

Note that $C_2^-(D_{10})$ is isomorphic to $C_2^-(D')$ via the removal of bivalent vertices, so our goal is to show that $C_2^-(D) \simeq_1 C_2^-(D_{10})$ as a filtered chain complex over $R(D')$. We will work over the larger ring $R(D')[U_3, U_4]$, but will not enforce linearity with respect to U_5 or U_6. First, note that $C_2^-(D_{00}) \cong C_2^-(D_{11})$. We see that we can apply the MOY II decomposition from Section 4.4.2 to write $C_2^-(D_{01}) = C_2^-(D_{11})\langle 1\rangle \oplus C_2^-(D_{00})\langle U_6\rangle$. We compute the maps induced by ϕ_1 and ϕ_2 on these decompositions to get an isomorphic cube of resolutions:

$$\begin{array}{ccc} C_2^-(D_{00}) & \xrightarrow{\quad 1 \quad} & C_2^-(D_{10}) \\ \Big\downarrow \begin{pmatrix} -U_3 \\ 1 \end{pmatrix} & & \Big\downarrow U_3 - U_2 \\ C_2^-(D_{11})\langle 1\rangle \oplus C_2^-(D_{00})\langle U_6\rangle & \xrightarrow{\begin{pmatrix} 1 & U_2 \end{pmatrix}} & C_2^-(D_{11}). \end{array}$$

This is the first of several times we will use Lemma A.2.1 to simplify a cube of resolutions in this chapter. This key lemma allows us to effectively cancel out isomorphisms of direct summands in cubes. In this case, it yields the E_1-quasi-isomorphic complex:

$$\begin{array}{ccc} 0 & \longrightarrow & C_2^-(D_{10}) \\ \Big\downarrow & & \Big\downarrow \\ 0 & \longrightarrow & 0\,. \end{array}$$

We conclude by noting that $C_2^-(D_{10}) \cong C_2^-(D')$ as chain complexes over $R(D')[U_3, U_4]$. This proves invariance under one type of Reidemeister II move; the proof of the mirror-image move is analogous.

□

4.5.2. Reidemeister III

We will prove invariance under the Reidemeister III move shown in Figure 4.5.2, which corresponds to sliding a strand over a positive crossing. In terms of the braid group, it represents the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. All other variations of the Reidemeister III move follow from this one plus the Reidemeister II invariance result from Theorem 4.5.1.

Theorem 4.5.2. *If D and D' are two partially singular braid diagrams that differ by a Reidemeister III move, then $C_2^-(D) \simeq_1 C_2^-(D')$.*

Proof. Let D be the diagram on the left and D' the diagram on the right in Figure 4.5.2. We will use Lemma A.2.1 to simplify $C_2^-(D)$ and $C_2^-(D')$ to see they

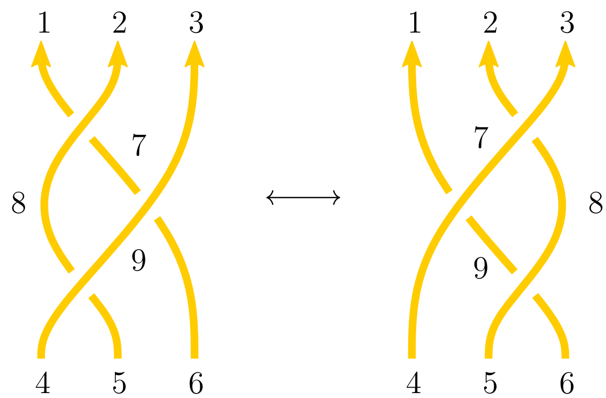
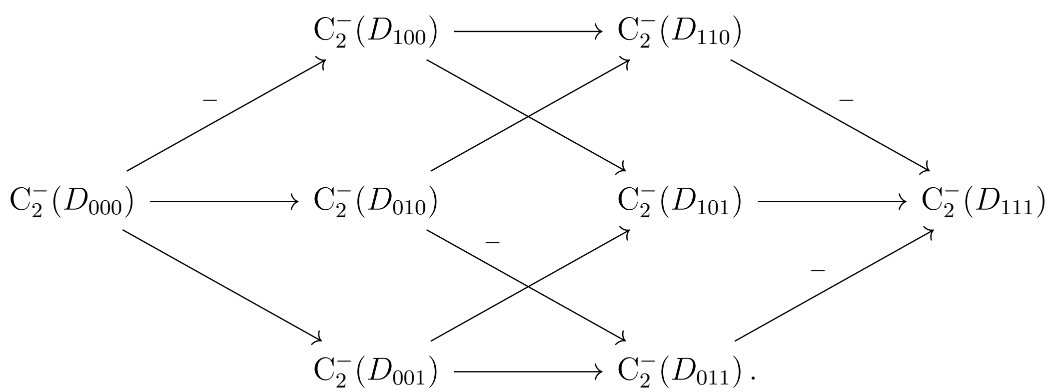


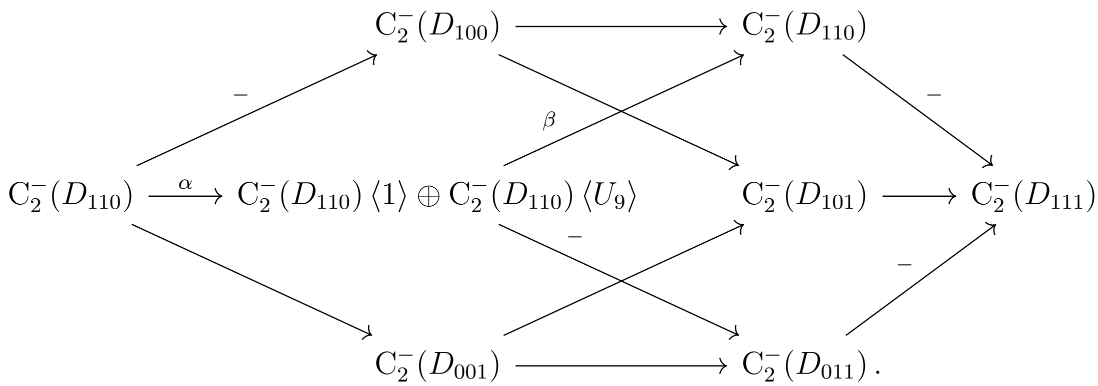
Figure 4.5.2: A Reidemeister III move.

have the same homotopy type. Label the edges of D with variables $U_1, \ldots, U_9$ as in Figure 4.5.2, and order the crossings from top to bottom. We expand the cube of resolutions for $\mathrm{C}_2^-(D)$ as:

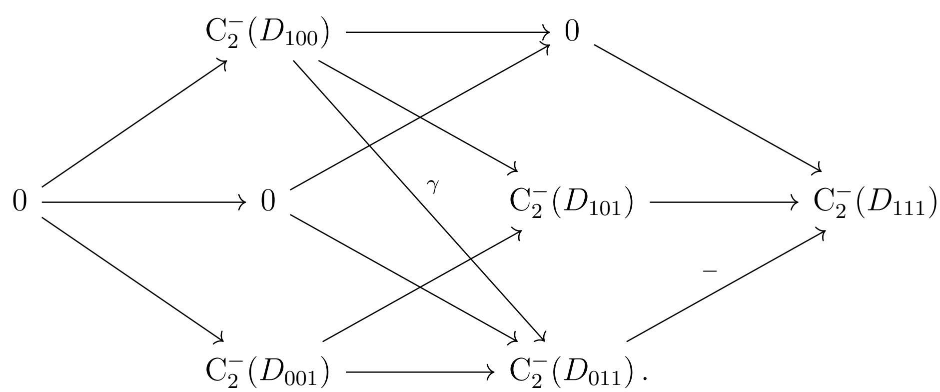


Since our local picture of D consists of only positive crossings, all edge maps in this cube are given by $\phi_+ = 1$ up to a sign assignment, which we take to be the one in the above cube of resolutions without loss of generality.

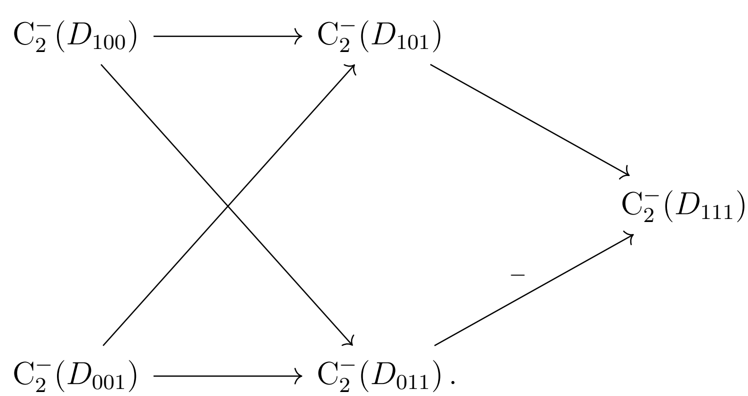
By Theorem 4.4.6, we note $\mathrm{C}_2^-(D_{000}) \cong \mathrm{C}_2^-(D_{110}) \oplus \Upsilon$, where Υ is acyclic. By a slight generalization of Lemma A.2.3, we get an E_1-quasi-isomorphic cube after replacing $\mathrm{C}_2^-(D_{000})$ by $\mathrm{C}_2^-(D_{110})$. Furthermore, Theorem 4.4.4 gives us that $\mathrm{C}_2^-(D_{010}) \cong \mathrm{C}_2^-(D_{110})\,\langle 1 \rangle \oplus \mathrm{C}_2^-(D_{110})\,\langle U_9 \rangle$. Therefore, the above cube is E_1-quasi-isomorphic to:



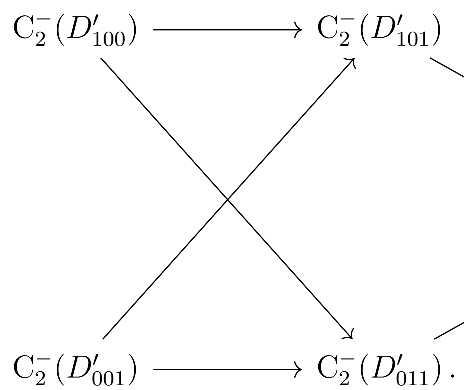
We compute the induced maps in the above cube to be $\alpha = \begin{pmatrix} -U_3 \\ 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & U_2 \end{pmatrix}$. By Lemma A.2.1, we can cancel the isomorphisms of direct summands in the above cube to obtain the E_1-quasi-isomorphic complex:



Removing the trivial complexes in the above cube, and noting that the map γ induced by cancellation is given by multiplication by 1, we get the complex:



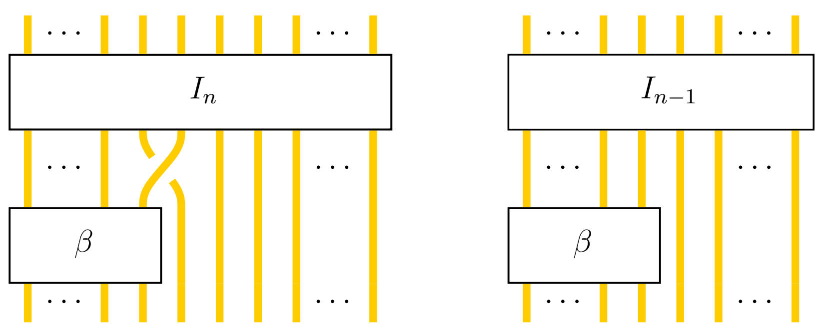
Now, recalling that we have a second diagram D' to work with, we may go through the same steps to simplify $\mathrm{C}_2^-(D')$ to get the complex:



We conclude by noting that the reduced complexes for $C_2^-(D)$ and $C_2^-(D')$ are isomorphic via the map that reflects the complexes about a horizontal axis, i.e. swaps the 100- and 001-resolutions, swaps the 101- and 011-resolutions, and fixes the 111-resolution. This map is a chain map since all the edge maps are ± 1, and therefore $C_2^-(D) \simeq_1 C_2^-(D')$.

□

4.5.3. Stabilization



D: Stabilized/Original D': Original/Destabilized

Figure 4.5.3: Diagrams related by a positive stabilization. Depending on context, we will either consider the diagram D and its *destabilization* D', or we will consider the diagram D' and its *stabilization* D.

Let $\beta \in B_{n-1}$ be an element of the braid group for $n \geq 2$, and let $\sigma_{n-1} \in B_n$ be the generator which introduces a positive crossing between strands $n-1$ and n. The *positive stabilization* of β is the braid $\sigma_{n-1}\beta \in B_n$, where here we are considering β

as an element of B_n via the natural inclusion $B_{n-1} \hookrightarrow B_n$. Analogously, the *negative stabilization* of β is the braid $\sigma_{n-1}^{-1}\beta$. For a braid $\beta = \sigma_{n-1}^{\pm 1}\beta' \in B_n$ in the image of one of these operations, we say that $\beta' \in B_{n-1}$ is the *destabilization* of β.

Theorem 4.5.3. *The E_1-homotopy type of the filtered complex $C_2^-(\beta)$ is invariant under positive and negative (de)stabilization, i.e. $C_2^-(\sigma_{n-1}\beta) \simeq_1 C_2^-(\sigma_{n-1}^{-1}\beta) \simeq_1 C_2^-(\beta)$*

We note that β, $\sigma_{n-1}\beta$, and $\sigma_{n-1}^{-1}\beta$ all have isotopic braid closures. Before we prove stabilization invariance, we will need to relate diagrams containing the open braid diagrams I_n and I_{n-1}, as the (de)stabilization operations alter the number of strands of our partially singular braid diagrams. Therefore, we first note that we can see I_{n-1} as a sub-diagram of I_n by ignoring the rightmost vertices in every row, as in Figure 4.5.4.

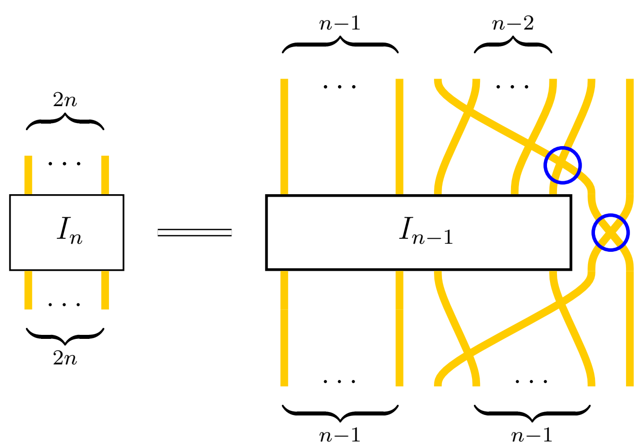


Figure 4.5.4: A recursive definition of I_n.

Consider $I_n(\sigma_{n-1}\beta)$, as shown in Figure 4.5.5. Let R be the polynomial ring over all edges not labeled in Figure 4.5.5, and label the rest of the edges accordingly, so that $R(I_n(\sigma_{n-1}\beta)) = R[U_1, U_2, U_3, U_4, U_5]$.

Let $\phi = \phi_+ = 1$ be the edge map corresponding to the positive crossing of σ_{n-1}. We write the one-dimensional cube of resolutions corresponding to resolving the crossing:

$$\mathrm{C}_2^-(I_n(\sigma_{n-1}\beta)_0) \xrightarrow{\phi} \mathrm{C}_2^-(I_n(\sigma_{n-1}\beta)_1).$$

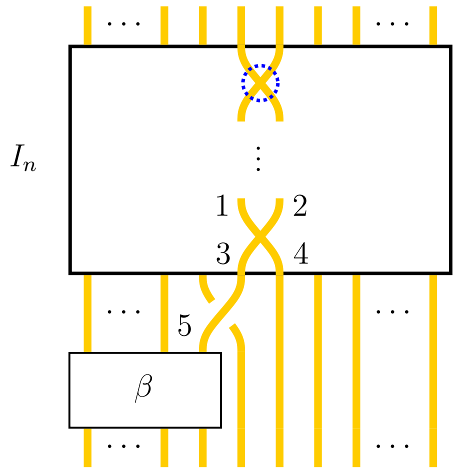


Figure 4.5.5: Relevant edge labels near σ_{n-1}. Note that the top vertex is fixed only when $n = 2$, and is otherwise free for $n \geq 3$.

The diagrams corresponding to these resolutions are illustrated in Figure 4.5.6.

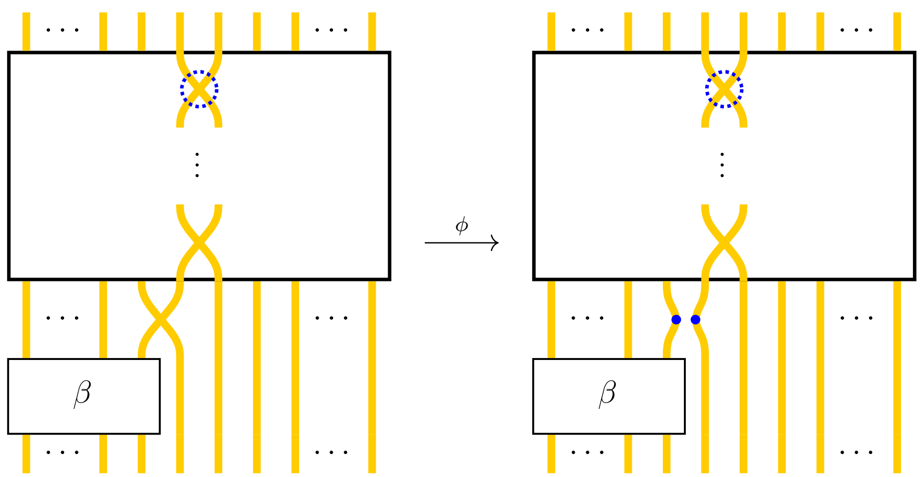


Figure 4.5.6: The mapping cone decomposition induced by σ_{n-1}.

Our goal now is to use MOY moves to modify both resolutions so that they can be represented using a common diagram, tracking the effect on the complexes. By an abuse of notation, we will denote this common diagram $I'_n(\beta)$, which is gotten analogously to $I_n(\beta)$: we place a straight strand to the right of β, place n straight strands to the right of that, top this diagram with I'_n, and take the braid closure. It remains to define I'_n. We let I'_n be I_n, without the singular vertex between strands n and $n+1$ in the first layer. As for I_n, we can also define I'_n by building on I_{n-1}, as in Figure 4.5.7.

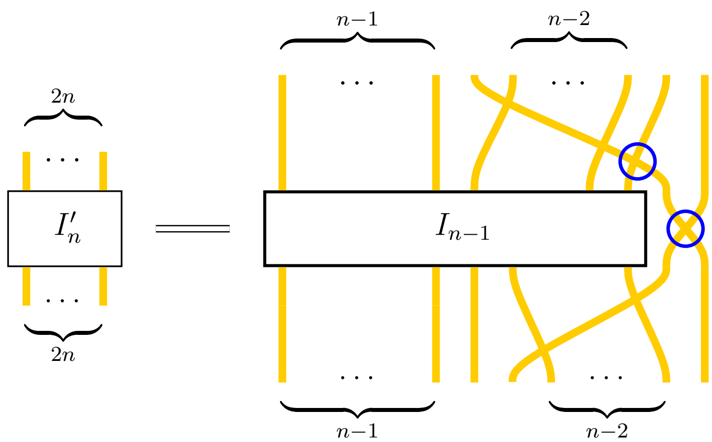


Figure 4.5.7: Building I'_n from I_{n-1}.

With this definition in mind, we now see that $I_n(\sigma_{n-1}\beta)_0$ is one MOY III move away from $I'_n(\beta)$, and $I_n(\sigma_{n-1}\beta)_1$ is one MOY II move away from $I'_n(\beta)$. On the one-dimensional cube of resolutions, then, we get

$$\mathrm{C}_2^-(I'_n(\beta))\,\langle U_3 - U_5\rangle \oplus \Upsilon \xrightarrow{\phi} \mathrm{C}_2^-(I'_n(\beta))\,\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\,\langle U_4\rangle\ .$$

Since Υ is acyclic, we can ignore it by Lemma A.2.2. We compute the map induced by ϕ on the summands as:

$$\mathrm{C}_2^-(I'_n(\beta))\,\langle U_3 - U_5\rangle \xrightarrow{\begin{pmatrix} U_1+U_2-U_5 \\ -1 \end{pmatrix}} \mathrm{C}_2^-(I'_n(\beta))\,\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\,\langle U_4\rangle\ .$$

Since the -1 entry represents an isomorphism of $\mathrm{C}_2^-(I'_n(\beta))$ summands, we can cancel it by Lemma A.2.1. This proves the following lemma.

Lemma 4.5.4. *The complexes $C_2^-(\sigma_{n-1}\beta)$ and $C_2^-(I'_n(\beta))$ have the same E_1-homotopy type.*

Therefore, to prove conjugation invariance, it remains to prove the following proposition.

Proposition 4.5.5. *The complexes $C_2^-(I_n'(\beta))$ and $C_2^-(\beta)$ have the same E_1-homotopy type.*

Proof. To begin, we note that $I_n'(\beta)$ and $I_{n-1}(\beta)$ are really braid closures, so for ease of understanding the upcoming MOY moves, we will replace our usual depiction of $I_n'(\beta)$ with a shifted version, as in Figure 4.5.8.

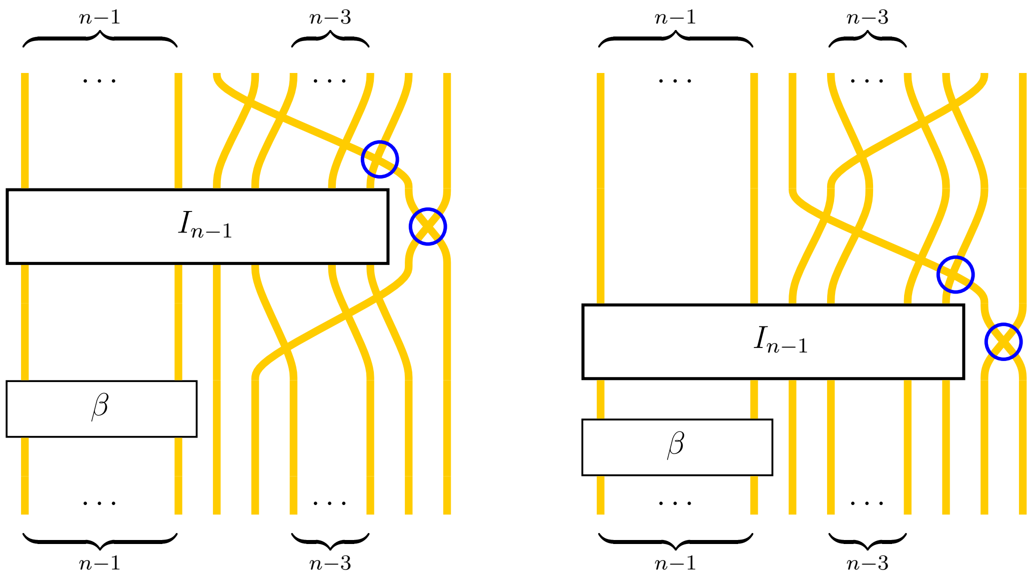


Figure 4.5.8: Shifting vertices in $I_n'(\beta)$.

In this shifted version, we identify the local picture on the left in Figure 4.5.9, consisting of a pair of intersecting strands and $n-2$ other strands which intersect both. We can apply $n-2$ MOY III moves to simplify this part of the diagram to the local picture on the right in Figure 4.5.9, consisting of $n-2$ straight strands and one pair of intersecting strands. By Theorem 4.4.6, each of these preserves the E_1-homotopy type of the complex. The global picture at this stage can be seen on the left in Figure 4.5.10. To arrive at the diagram for $I_{n-1}(\beta)$, we apply two MOY I moves to the two remaining fixed vertices outside of I_{n-1}. By Theorem 4.4.1, each of these preserves the E_1-homotopy type of the complex. This leads to the diagram on the right in Figure 4.5.10, which is exactly the diagram for $I_{n-1}(\beta)$.

□

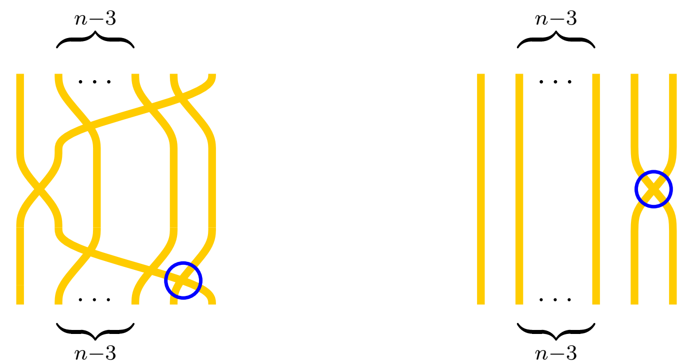


Figure 4.5.9: Local pictures of diagrams related by a sequence of $n-2$ MOY III moves.

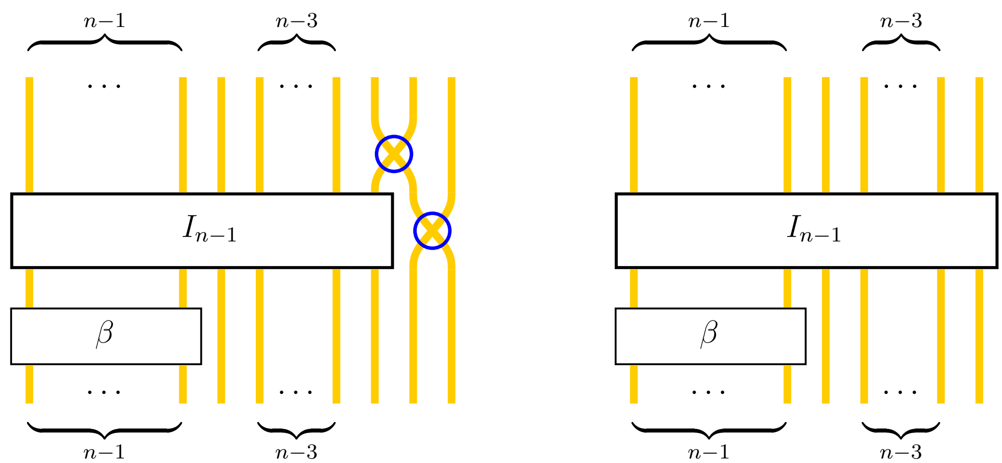


Figure 4.5.10: The last step in simplifying $I'_n(\beta)$ in the proof of Proposition 4.5.5.

Proof of Theorem 4.5.3. We have shown that $\mathrm{C}_2^-(\sigma_{n-1}\beta) \simeq \mathrm{C}_2^-(I'_n(\beta)) \simeq \mathrm{C}_2^-(\beta)$. We can simplify $\mathrm{C}_2^-(\sigma_{n-1}^{-1}\beta)$ to $\mathrm{C}_2^-(I'_n(\beta))$ as well. Using the same edge labels and notation as before, we write the cube of resolutions for $\mathrm{C}_2^-(\beta)$ as

$$\mathrm{C}_2^-(I_n(\beta)_0) \xrightarrow{\quad\phi\quad} \mathrm{C}_2^-(I_n(\beta)_1)$$

where this time $\phi = \phi_- = U_3 - U_5$. Applying an MOY II move to $I_n(\beta)_0$ and an MOY III move to $I_n(\beta)_1$ to write our complexes in terms of $\mathrm{C}_2^-(I'_n(\beta))$ gives us the complex

$$\mathrm{C}_2^-(I'_n(\beta))\,\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\,\langle U_4\rangle \xrightarrow{\quad\phi\quad} \mathrm{C}_2^-(I'_n(\beta))\,\langle U_3 - U_5\rangle \oplus \Upsilon\,.$$

Again, excluding Υ and computing the map induced by ϕ, we get:

$$\mathrm{C}_2^-(I'_n(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\langle U_4\rangle \xrightarrow{\begin{pmatrix}1 & U_4\end{pmatrix}} \mathrm{C}_2^-(I'_n(\beta))\langle U_3 - U_5\rangle\ .$$

As before, we may cancel the 1 in the above matrix to see that $\mathrm{C}_2^-(\sigma_{n-1}^{-1}\beta) \simeq_1 \mathrm{C}_2^-(I'_n(\beta))$ as well. The rest of the proof follows from Proposition 4.5.5. Since we have covered both the positive and negative cases, this suffices to show invariance under stabilization. □

4.5.4. Conjugation

Conjugation invariance is the following statement:

Theorem 4.5.6. *For any $\alpha, \beta \in B_n$, we have that $C_2^-(\alpha^{-1}\beta\alpha) \simeq_1 C_2^-(\beta)$.*

To begin, we prove a lemma relating complexes associated to diagrams that locally look like the pictures in Figure 4.5.11.

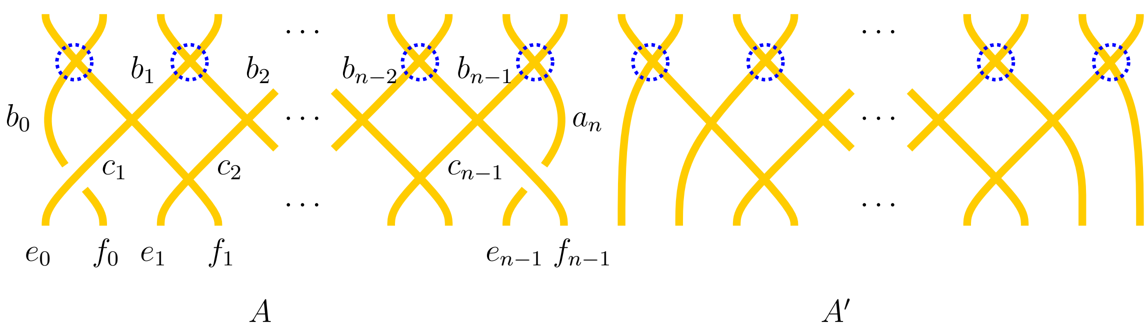


Figure 4.5.11: Local pictures of diagrams with equivalent $\mathrm{C}_2^-(-)$.

Lemma 4.5.7. *Let A and A' be partially singular braid diagrams that are identical outside of a specific region, where they look like the diagrams in Figure 4.5.11, i.e. A has two opposite crossings whereas A' has oriented smoothings. Then $C_2^-(A) \simeq_1 C_2^-(A')$.*

Proof. Let $\phi_1 = \phi_+ = 1$ be the edge map corresponding to the left (positive) crossing, and let $\phi_2 = \phi_- = a_n - e_{n-1}$ be the edge map corresponding to the right (negative) crossing. We expand the cube of resolutions for $\mathrm{C}_2^-(A)$ as follows:

$$\begin{array}{ccc} \mathrm{C}_2^-(A_{00}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2} \\ \mathrm{C}_2^-(A_{01}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A_{11})\,. \end{array}$$

The diagrams for these four partial resolutions look like Figure 4.5.12.

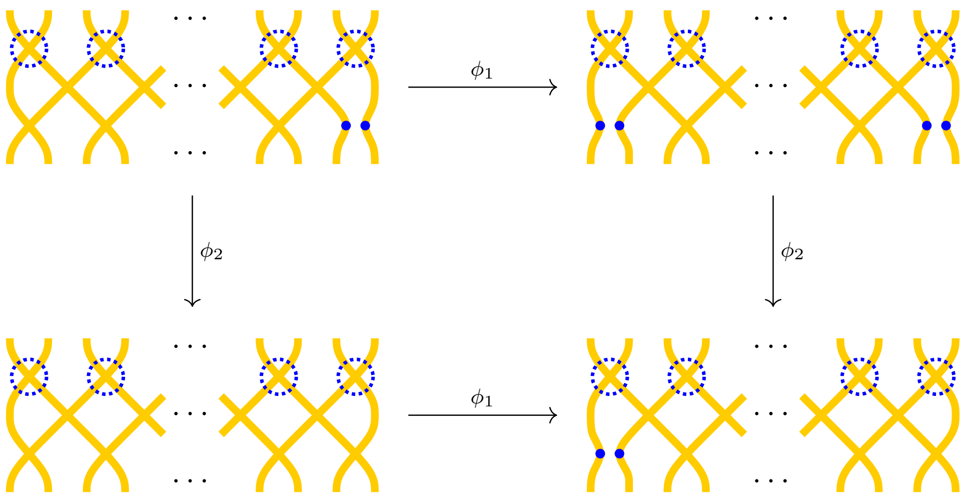


Figure 4.5.12: The cube of resolutions for diagram A in Figure 4.5.11.

We can use MOY III moves to simplify three of the four corners of this cube. For A_{00}, we can start with a MOY III move on the left, simplifying the diagram. Each MOY III move we do allows us to do another, until we have done $n-1$ such moves moving left-to-right. We denote the resulting diagram A''_{00}; it is shown in Figure 4.5.13. By Theorem 4.4.6, A_{00} and A'_{00} are E_1-quasi-isomorphic.

Similarly, we can simplify A_{11} to A'_{11} by performing $n-1$ MOY III moves right-to-left, and we can simplify A_{01} to A'_{01} by performing $n-1$ MOY III moves left-to-right. In each case, Theorem 4.4.6 ensures we are preserving the E_1-quasi-isomorphism type.

The resulting diagrammatic cube of resolutions is shown in Figure 4.5.13. Thus we obtain the complex

$$\begin{array}{ccc} \mathrm{C}_2^-(A''_{00})\langle x\rangle & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2} \\ \mathrm{C}_2^-(A''_{01})\langle x\rangle \oplus \Upsilon & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A''_{11})\langle x\rangle\ . \end{array}$$

This cube ignores the Υ summands in A_{00} and A_{11} by Lemma A.2.3, but retains the $\Upsilon \coloneqq \Upsilon_1 \oplus \ldots \oplus \Upsilon_{n-1}$ summand in A_{01}. Further, as every vertex in the middle row is free, we may choose $x = (b_1 - c_1)\ldots(b_{n-1} - c_{n-1})$ to be the generator for all three complexes modulo the linear ideal L.

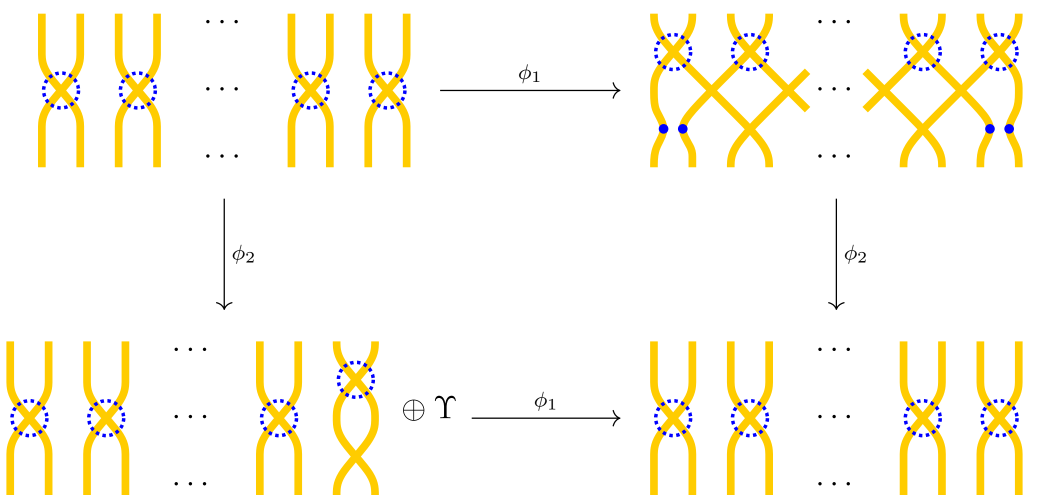


Figure 4.5.13: The reduced cube of resolutions.

We further decompose A''_{01} via an MOY II move on the right into two copies of A''_{00}, generated by x and $a_n x$. Additionally, we see that A''_{00} and A''_{11} are isomorphic. We can compute the maps induced by ϕ_1 and ϕ_2 and write our complex as

$$\begin{array}{ccc} \mathrm{C}_2^-(A''_{00})\langle x\rangle & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A_{10}) \\ \downarrow{\scriptstyle \begin{pmatrix} -e_{n-1} \\ 1 \\ * \end{pmatrix}} & & \downarrow{\scriptstyle \phi_2} \\ \mathrm{C}_2^-(A''_{00})\langle x\rangle \oplus \mathrm{C}_2^-(A''_{00})\langle a_n x\rangle \oplus \Upsilon & \xrightarrow{\begin{pmatrix} 1 & f_{n-1} & * \end{pmatrix}} & \mathrm{C}_2^-(A''_{00})\langle x\rangle\ . \end{array}$$

We may cancel the 1s in the above matrices to reduce the complex by Lemma A.2.1 to obtain

$$\begin{array}{ccc} 0 & \longrightarrow & \mathrm{C}_2^-(A_{10}) \\ \downarrow & & \downarrow \\ \Upsilon & \longrightarrow & 0\,. \end{array}$$

Since Υ is a direct sum of E_1-acyclic complexes, we see that the E_2-page of the above complex is isomorphic to that of $\mathrm{C}_2^-(A_{10})$, which is isomorphic to $\mathrm{C}_2^-(A')$, thereby proving Lemma 4.5.7 in the case of a positive crossing on the left and a negative one on the right.

The opposite case is analogous; applying the same moves (mirrored horizontally) results in the complex

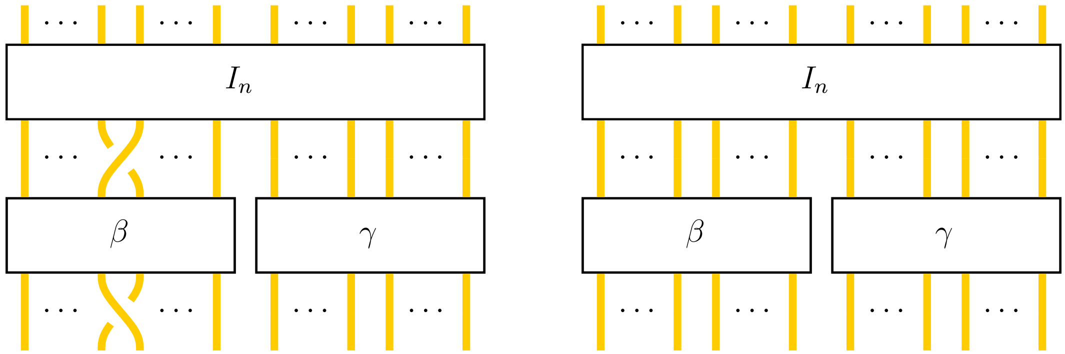
$$\begin{array}{ccc} \mathrm{C}_2^-(A''_{00})\,\langle x\rangle & \xrightarrow{\begin{pmatrix} f_0 \\ -1 \\ * \end{pmatrix}} & \mathrm{C}_2^-(A''_{00})\,\langle x\rangle \oplus \mathrm{C}_2^-(A''_{00})\,\langle b_0x\rangle \oplus \Upsilon \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & e_0 & * \end{pmatrix}} \\ \mathrm{C}_2^-(A_{01}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A''_{00})\,\langle x\rangle \end{array}$$

which we can simplify to get that the E_2-page is the same as that of $\mathrm{C}_2^-(A_{01})$ and therefore $\mathrm{C}_2^-(A')$. □

With this lemma in hand, we are now prepared to prove conjugation invariance.

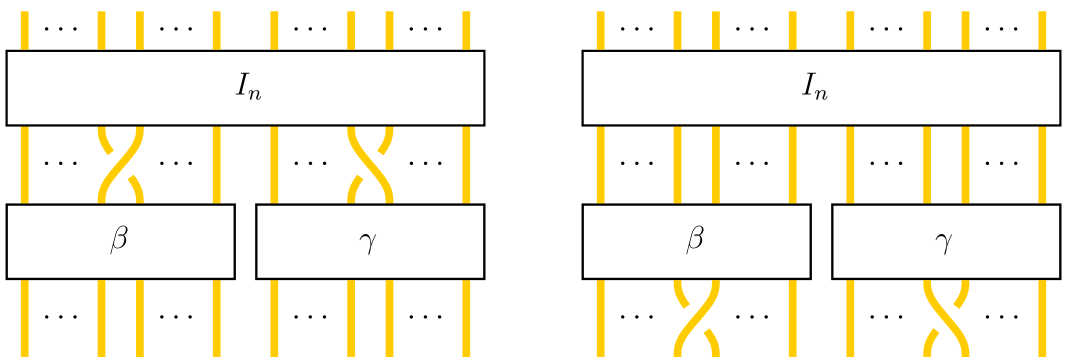
Proof of Theorem 4.5.6. It suffices to prove this in the case that $\alpha = \sigma_i^{\pm 1}$ is any generator of the braid group (or its inverse). Therefore, let $\sigma_i \in B_n$ be the generator which introduces a positive crossing between strands i and $i+1$.

Graphically, we would like to show that $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$, where D and D' are the partially singular braids depicted in Figure 4.5.14, when $\gamma = 1$. In order to prove



D D'

Figure 4.5.14: When $\gamma = 1$, the diagram D is the result of conjugating the braid β in D' by a generator of the braid group.



D'' D'''

Figure 4.5.15: Alternate diagrams for proving conjugation invariance.

this, we will instead show that $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D') \simeq_1 \mathrm{C}_2^-(D''')$ for a generic $\gamma \in B_n$, where D'' and D''' are the diagrams in Figure 4.5.15.

Since we are considering the case $\alpha = \sigma_i$, note that in D'', the positive crossing occurs between strands i and $i+1$. If we decompose I_n, we see that for any i, we locally get a picture like Figure 4.5.11, where the top row of vertices is fixed if $i = 1$, and free if $i > 1$. Therefore, we may apply Lemma 4.5.7 directly to see that $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D')$. Additionally, note that in D''', the negative crossing occurs between strands i and $i+1$. If we decompose I_n, we see that for any i, we locally get a picture like Figure 4.5.11, except that the bottom row of vertices is fixed if

$i = 1$, and free if $i > 1$. In the latter case, this is not an issue and we may proceed as before to use Lemma 4.5.7 to prove that $\mathrm{C}_2^-(D''') \simeq_1 \mathrm{C}_2^-(D')$. If $i = 1$, then we first use Theorem 4.3.1 to relabel the top row of vertices as free and the bottom row as fixed; this diagram is still in $\mathcal{D}^{\mathcal{R}}$ as it contains the open braid S_{2n} from [14] as a sub-diagram, so we may proceed with the rest of the proof as usual.

Therefore, we can prove the desired equivalence $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$ when $\gamma = 1$ by first performing a Reidemeister II move to add two crossings to the right side of D, then using the equivalences $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D')$ and $\mathrm{C}_2^-(D''') \simeq_1 \mathrm{C}_2^-(D')$ to simplify the diagram to D'. This proves Theorem 4.5.6 in the case of $\alpha = \sigma_i$. We illustrate these steps for the case $n = 2$ and $\alpha = \sigma_1$ in Figure 4.5.16. The proof for $\alpha = \sigma_i^{-1}$ is analogous, from which the proof for general $\alpha \in B_n$ follows. □

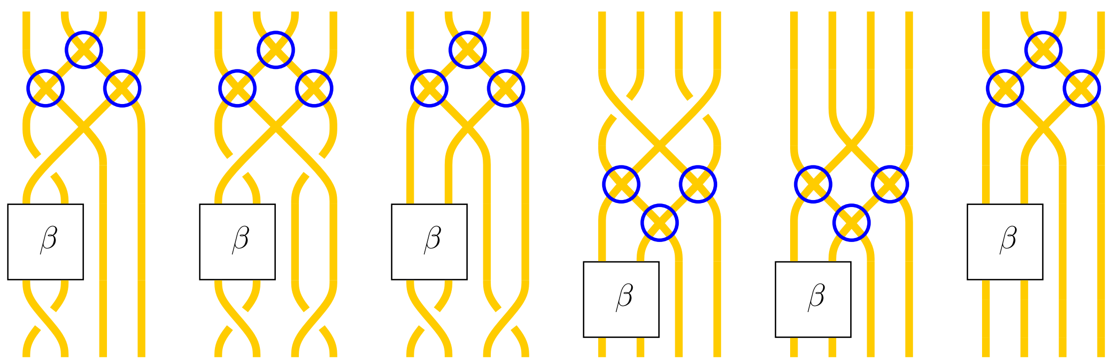


Figure 4.5.16: The steps to prove conjugation invariance for $n = 2$ and $\alpha = \sigma_1$.

Appendix A

Appendix

Section A.1

$\mathbb{Z}$-filtrations

In the process of recalling annular Khovanov homology in Section 3.2, we need to construct a filtration on the Khovanov complex. Our homology in Section 3.4 is also defined using a more general notion of $\mathbb{Z}^n$-filtration. We review the theory of filtrations by $\mathbb{Z}$ (which we may call $\mathbb{Z}$-filtrations) below.

The usual notion of a *(bounded, ascending) filtration* $\mathcal{F}$ of an R-module M is a sequence of submodules:

$$\ldots \subseteq \mathcal{F}_{-1}M \subseteq \mathcal{F}_0 M \subseteq \mathcal{F}_1 M \subseteq \ldots$$

such that $\mathcal{F}_i M = 0$ and $\mathcal{F}_j M = M$ for some $i, j \in \mathbb{Z}$. Additionally, we require that $\mathcal{F}_s M = 0$ and $\mathcal{F}_t M = M$ for some $s, t \in \mathbb{Z}$.

A module M equipped with a filtration $\mathcal{F}$ is a *filtered module* $(M, \mathcal{F})$. We often overload notation by referring to filtered modules by their underlying modules. If two modules M and N are filtered, then a map $f : M \to N$ is *filtered* if $f(\mathcal{F}_a M) \subseteq \mathcal{F}_a N$

for all $a \in \mathbb{Z}$. A *filtered isomorphism* is a filtered map with a filtered inverse.

Given a filtered module M and an injective map of modules $f : N \hookrightarrow M$, define the *induced filtration* on N to be the filtration $\mathcal{F}'_a N = f^{-1}(\mathcal{F}_a M)$ for $a \in \mathbb{Z}$. Oftentimes, $N \hookrightarrow M$ will be thought of as an inclusion of a submodule. Similarly, given a filtered module M and a surjective map of modules $f : M \twoheadrightarrow N$, define the *quotient filtration* on N to be the filtration $\mathcal{F}'_a N = f(\mathcal{F}_b M)$ for $b \in \mathbb{Z}$. Oftentimes, $M \twoheadrightarrow N$ will be thought of as a quotient by a submodule.

Associated graded objects. Given a filtered module M, we define the *associated graded module* $\mathrm{gr}(M)$ as follows:

$$\mathrm{gr}(M) := \bigoplus_{a \in \mathbb{Z}} \mathrm{gr}_a(M)$$
$$\mathrm{gr}_a(M) := \frac{\mathcal{F}_a M}{\mathcal{F}_{a-1} M}$$

For a vector space V, it is easy to see that $V \cong \mathrm{gr}(V)$ as modules, as one can show that a basis for $\mathrm{gr}(V)$ induces a basis for V. This is not true for modules over general rings. For example, the filtration of $\mathbb{Z}$ as a module over itself given by $0 \subset 2\mathbb{Z} \subset \mathbb{Z}$ has associated graded module isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Because it is in the proof of Theorem 3.1.1, we define a *filtered projective* module M to be a filtered module such that $\mathrm{gr}_a(M)$ is projective for all $a \in \mathbb{Z}$.

$\mathbb{Z}$-filtered complexes. A *filtration of a chain complex* (C, ∂) is a filtration of C that respects ∂ in the sense that $\partial(\mathcal{F}_a C) \subseteq \mathcal{F}_a C$ for $a \in \mathbb{Z}$. A *filtered chain complex* is a chain complex (C, ∂) equipped with a filtration $\mathcal{F}$; we will often overload notation by simply referring to $(C, \partial, \mathcal{F})$ as C. A *filtered chain map* is a map of chain complexes that also respects the filtration, i.e. a map of modules $f : C \to D$ that commutes with ∂ and "commutes" with $\mathcal{F}$. Similarly, the *associated graded complex* $\mathrm{gr}(C, \partial, \mathcal{F}) =$

$(\mathrm{gr}(C), \mathrm{gr}(\partial))$ has underlying module the associated graded module and differential induced by the quotient operation. Even for chain complexes of vector spaces, we no longer have that $C \cong \mathrm{gr}(C)$ as chain complexes.

A *quasi-isomorphism* is a map $f : C \to D$ of chain complexes that induces an isomorphism $f_* : H_*(C) \to H_*(D)$. A *filtered quasi-isomorphism* is a filtered chain map $f : C \to D$ that induces quasi-isomorphisms $\mathrm{gr}_a(f) : \mathrm{gr}_a(C) \to \mathrm{gr}_a(D)$ for all $a \in \mathbb{Z}$. Note that this is a stronger condition than just asking that f is a filtered map and also a quasi-isomorphism. The class of filtered quasi-isomorphisms includes filtered isomorphisms, and the composition of two filtered quasi-isomorphisms is again a filtered quasi-isomorphism.

A filtered map $f : C \to D$ is said to be *strict* if $f(\mathcal{F}_a C) = f(C) \cap \mathcal{F}_a D$ (not just $\subseteq$) for all $a \in \mathbb{Z}$. Essentially by definition, inclusions of filtered submodules and projections onto filtered quotient modules are always strict. The following fact about strict maps of filtered complexes will be useful:

Lemma A.1.1. *If $f : C \to D$ is a strict map of filtered complexes, then the following sequence is exact:*

$$0 \longrightarrow gr(\ker f) \longrightarrow gr(C) \xrightarrow{gr(f)} gr(D) \longrightarrow gr(\operatorname{coker} f) \longrightarrow 0$$

Specifically, if f is injective (surjective), then $\ker f$ *(respectively,* $\operatorname{coker} f$*) is zero, so the above gives us a short exact sequence of chain complexes.*

A proof of the above lemma can be found at [51, Section 0120].

$\mathbb{Z}$-filtered vector spaces. When we are working over a field, we can make some convenient simplifications. A filtered vector space V is isomorphic to its associated graded vector space $\mathrm{gr}(V)$. Therefore, a $\mathbb{Z}$-filtration $\mathcal{F}$ of a vector space V is equivalent to a $\mathbb{Z}$-grading g of V. Under this correspondence, filtered isomorphisms and graded isomorphisms coincide. The following lemmas explain how these gradings interact with the induced and quotient filtrations described earlier.

Lemma A.1.2. *Let V be a filtered vector space with grading g, and let $f : W \to V$ be an injective linear map. Give W the induced filtration; the associated grading on a homogeneous element $w \in W$ is then given by $\mathrm{g}(f(w))$.*

Lemma A.1.3. *Let V be a filtered vector space with grading g, and let $f : V \to W$ be a surjective linear map. Give W the quotient filtration; the associated grading on a homogeneous element $w \in W$ is then given by the minimum of $\mathrm{g}(v)$ over all homogeneous elements $v \in f^{-1}(w)$.*

Section A.2

Reduction

As many topological invariants take values in homotopy types of chain complexes and similar structures, it is helpful to be able to simplify presentations for these objects. In this section, we review some common techniques for "reducing" complexes, i.e. finding weakly equivalent complexes with a smaller dimension.

We note that all four lemmas in this section are true for filtered complexes, replacing maps with filtered maps and quasi-isomorphisms with filtered quasi-isomorphisms. Additionally, if we instead assume that filtered maps have filtration degree 1, then these lemmas still hold, replacing $\mathrm{cone}(f)$ with $\mathrm{cone}_1(f)$ and filtered quasi-isomorphism with E_1-quasi-isomorphism.

A.2.1. Gaussian elimination

This first technique is very general, and was originally stated for additive categories in [8].

Lemma A.2.1. *[8, Lemma 4.2] If $\varphi : A \to B$ is an isomorphism of complexes, then the double complexes*

$$C \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} A \oplus D \xrightarrow{\begin{pmatrix}\varphi & \delta\\ \gamma & \epsilon\end{pmatrix}} B \oplus E \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} F$$

and

$$C \xrightarrow{\beta} D \xrightarrow{\epsilon-\gamma\varphi^{-1}\delta} E \xrightarrow{\nu} F$$

are quasi-isomorphic.

Specialized versions of this lemma can be found in papers from various subject areas and various times. For example, if we specialize to the case that we have complexes of free modules over a ring R, and assume that $A \cong B \cong R^1$, we may choose a basis $\{a\}$ of A and $\{b\}$ of B such that $\phi(a) = rb$ for some $r \in R$. The condition that ϕ is an isomorphism is then precisely the assertion that $r \in R^\times$ is a unit. This form of Lemma A.2.1 is the chain complex reduction algorithm introduced in [20]. This appears in knot theory literature as well in [46].

It can sometimes be helpful to visualize chain complexes as directed graphs. Given a free chain complex C over R, we can choose a basis $\beta = \{b_1, \ldots, b_n\}$ to represent C as an edge-labeled directed graph G. The vertices $V(G) = \beta$ are the basis elements, and the edge $b_i \to b_j$ is labeled with $r \in R$ if and only if b_j appears with coefficient r when $\partial(b_i)$ is written in the basis β; if $r = 0$, we often want to exclude the edge from the graph. This motivates why this technique (and its generalization to A_∞-algebras is sometimes referred to as an "edge reduction" algorithm, as in [29]. Further generalizations to the cases of Type D and Type DA structures are documented in [55], for example.

A.2.2. Acyclic subobjects and quotients

One of the most basic reduction tools for chain complexes is identification of acyclic subcomplexes or quotients. For example, if C is a chain complex with a subcomplex $C' \subseteq C$ such that $H_*(C') \cong 0$, then the existence of the quotient map $C \to C/C'$ along with the long exact sequence in homology tells us that $C \simeq C/C'$. Similarly, if instead we know that C/C' is acyclic, then the inclusion $C' \to C$ induces the equivalence $C \simeq C'$.

If our complex looks like a mapping cone, we can exploit this structure to eliminate submodules that may not even be subcomplexes.

Lemma A.2.2. *Let $f : A \to B$ be a map of complexes, and suppose that $A \cong A' \oplus A''$ and $B \cong B' \oplus B''$, where A'' and B'' are acyclic. Let $\iota : A' \hookrightarrow A$ and $\pi : B \twoheadrightarrow B'$ be the associated inclusion and projection maps, respectively. Then* $\operatorname{cone}(f) \simeq \operatorname{cone}(\pi \circ f \circ \iota)$.

Proof. First, we note that $\operatorname{cone}(\iota)$ is acyclic. One way to see this is via a cancellation argument: we have that $\operatorname{cone}(\iota) \cong (A' \to A' \oplus A'')$, which is quasi-isomorphic to A'' by Lemma A.2.1. Similarly, we get that $\operatorname{cone}(\pi)$, being quasi-isomorphic to B'', is acyclic as well.

For any two maps $\alpha : X \to Y$ and $\beta : Y \to Z$ of complexes, we have a long exact sequence relating the homology groups of $\operatorname{cone}(\alpha)$, $\operatorname{cone}(\beta)$, and $\operatorname{cone}(\beta \circ \alpha)$ (for example, via the octahedral axiom for triangulated categories applied to the derived category of R-modules). Therefore, we get that $\operatorname{cone}(f) \simeq \operatorname{cone}(f \circ \iota) \simeq \operatorname{cone}(\pi \circ f \circ \iota)$. □

Furthermore, if our complex looks like a cube of resolutions, we can use the iterated mapping cone structure to simplify the "source" and "sink" components.

Lemma A.2.3. *Let A, B, C, D be complexes, and suppose that $A \cong A' \oplus A''$ and $D \cong D' \oplus D''$, where A'' and D'' are acyclic. Let $\iota : A' \hookrightarrow A$ and $\pi : D \twoheadrightarrow D'$ be the* *associated inclusion and projection maps, respectively. Then the following two cube of resolutions complexes have the same homotopy type:*

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ \big\downarrow{\scriptstyle g_1} & & \big\downarrow{\scriptstyle g_2} \\ C & \xrightarrow{f_2} & D \end{array} \qquad\qquad \begin{array}{ccc} A' & \xrightarrow{f_1\circ\iota} & B \\ \big\downarrow{\scriptstyle g_1\circ\iota} & & \big\downarrow{\scriptstyle \pi\circ g_2} \\ C & \xrightarrow{\pi\circ f_2} & D' \end{array}.$$

Proof. We know that the inclusion $\mathrm{cone}(g_1 \circ \iota) \hookrightarrow \mathrm{cone}(g_1)$ and the projection $\mathrm{cone}(g_2) \twoheadrightarrow \mathrm{cone}(\pi \circ g_2)$ are quasi-isomorphisms by the proof of Lemma A.2.2.

$$\begin{array}{ccc} A' & \xrightarrow{\iota} & A \\ \big\downarrow{\scriptstyle g_1\circ\iota} & & \big\downarrow{\scriptstyle g_1} \\ C & \xrightarrow{\mathrm{id}_C} & C \end{array} \qquad\qquad \begin{array}{ccc} B & \xrightarrow{\mathrm{id}_B} & B \\ \big\downarrow{\scriptstyle g_2} & & \big\downarrow{\scriptstyle \pi\circ g_2} \\ D & \xrightarrow{\pi} & D' \end{array}$$

We can also view the maps $f_1 : A \to B$ and $f_2 : C \to D$ as components of a map $f : \mathrm{cone}(g_1) \to \mathrm{cone}(g_2)$. Therefore, we can compose f with the inclusion and projection to get a single map $f' : \mathrm{cone}(g_1 \circ \iota) \to \mathrm{cone}(\pi \circ g_2)$. By the same long exact sequence logic as before, the cone of this map has the same homotopy type as f, i.e.

$$\mathrm{cone}(f') = \mathrm{cone}(\mathrm{cone}(g_1 \circ \iota) \to \mathrm{cone}(\pi \circ g_2)) \simeq \mathrm{cone}(\mathrm{cone}(g_1) \to \mathrm{cone}(g_2)) = \mathrm{cone}(f)$$

We conclude by noting that the complex on the left in Lemma A.2.3 is $\mathrm{cone}(f)$, and the complex on the right is $\mathrm{cone}(f')$. □

While the above lemma is phrased only for squares, it can be iterated to reduce summands of higher-dimensional cubes as well.

A.2.3. Mapping Cones

Since our complexes in this thesis are often constructed as mapping cones, it will help to know when a quasi-isomorphism is induced by maps on the components of the cone.

Lemma A.2.4. *Suppose that we have the following commutative diagram of chain maps.*

$$\begin{array}{ccc} A_0 & \xrightarrow{g} & A_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ B_0 & \xrightarrow{g'} & B_1 \end{array}$$

Let $A = \operatorname{cone}(g)$ *and* $B = \operatorname{cone}(g')$*, so that we get a map* $f : A \to B$ *with components* f_0 *and* f_1*. If* f_0 *and* f_1 *are quasi-isomorphisms, then so is* f*.*

Proof. By properties of the mapping cone, f induces a map of short exact sequences (with grading shifts suppressed)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B & \longrightarrow & B_0 & \longrightarrow & 0 \end{array}$$

We can look at the induced map of long exact sequences in homology

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H_*(A_1) & \longrightarrow & H_*(A) & \longrightarrow & H_*(A_0) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle H_*(f_1)} & & \downarrow{\scriptstyle H_*(f)} & & \downarrow{\scriptstyle H_*(f_0)} & & \\ \dots & \longrightarrow & H_*(B_1) & \longrightarrow & H_*(B) & \longrightarrow & H_*(B_0) & \longrightarrow & \dots \end{array}$$

to conclude that $H_*(f)$ must be an isomorphism as well, so f is a quasi-isomorphism. □

To prove the filtered generalizations of Lemma A.2.4, one replaces $H_*(-)$ with $E_1(-)$ or $E_2(-)$ (see [52, Exercise 5.4.4]).

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