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Jones Polynomial Obstructions for Positivity of Knots

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JONES POLYNOMIAL OBSTRUCTIONS FOR POSITIVITY OF KNOTS

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Abstract

The fundamental problem in knot theory is distinguishing one knot from another. We accomplish this by looking at knot invariants. One such invariant is positivity. A knot is positive if it has a diagram in which all crossings are positive. A knot is almost-positive if it does not have a diagram where all crossings are positive, but it does have a diagram in which all but one crossings are positive. Given a knot with an almost-positive diagram, it is in general very hard to determine whether it might also have a positive diagram. This work provides positivity obstructions for three classes of knots that are distinguished by the second coefficient of their Jones polynomial, and we present three infinite families of examples of almost-positive knots whose non-positivity can be proved using the obstructions developed here.

Preface

Thank you to my advisor, Vladimir Chernov, and my committee members: Patricia Cahn, Peter Doyle, and Dana Williams.

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Contents

Chapter 1

Introduction

A *knot* is a smooth embedding of S^1 into $\mathbb{R}^3$. A *link* is a smooth embedding of n disjoint copies of S^1 into $\mathbb{R}^3$. Both knots and links are considered up to ambient isotopy. Each of the n embedded copies of S^1 is called a *component* of the link. We can study links by studying their *diagrams*, projections of the link onto $\mathbb{R}^2$ with isolated transverse double points, equipped with overcrossing and undercrossing information.

The main problem of knot theory is distinguishing one knot from another. We accomplish this by looking at knot invariants, and proving that one knot has properties that the other does not have. Many of the most powerful knot invariants are diagram-independent, meaning that any two diagrams of a knot will demonstrate the same property. More elusive are the diagram-existential properties, the ones that ask if there exists a diagram of a given knot that exhibits some special feature.

One such diagram-existential property is positivity. When a knot has a positive diagram, it forces many restrictions on the polynomial knot invariants. However, these restrictions are not enough to classify positive knots, as there are many non-positive knots that possess those same properties. The most difficult to distinguish are positive knots and almost-positive knots.

Not only are there many almost-positive knots that cannot be detected by these positivity obstructions, but in fact (as observed in [5]), most of the main results we have on positive knots also are true for almost-positive knots: positive and almost-positive knots both have positive Conway polynomials (Cromwell, [4]), have negative signatures (Przytycki and Taniyama, [14]), and are strongly quasipositive (Feller and Lobb, [5]). In general, it is very hard to distinguish positive knots from almost-positive knots.

The goals of this thesis, then, are to further the understanding of properties of positive link diagrams, and to develop these into positivity obstructions.

Chapter 2 contains background on the fundamental knot polynomials, with particular attention to the Kauffman state-sum model of the Jones polynomial.

Chapter 3 introduces the concept of balanced diagrams (of type 0), a special kind of positive link diagram with key property that the number of link components is equal to the number of circles in the all-B-smoothing Kauffman state. This allows us to construct a new bound on the maximum degree of the Jones polynomial of fibered positive links, links which are characterized by having their second Jones coefficient equal to 0. This appears on the arXiv in a slightly different form [2].

Chapter 4 expands the definition of balanced diagrams to include positive links with second Jones coefficient equal to ± 1 or ± 2, and we develop similar bounds on the maximum degree of their Jones polynomials as well.

We see in Chapter 3 that the result for fibered positive links allows us to complete the positivity classification of all knots up through crossing number 12, by showing that the seven remaining positivity-unknown 12-crossing knots are not positive. And we see at the end of Chapter 4 that the positivity obstructions developed in this work for the cases of positive links with second Jones coefficient equal to $0, \pm 1$, or ± 2 can each be used to identify infinitely many knots as almost-positive knots.

Chapter 2

Background

Section 2.1

Knot Polynomials

Knot polynomials are among many invariants that knot theorists use to distinguish knots from one another. The first knot polynomial developed was the Alexander polynomial $\Delta(t)$, developed by Alexander in 1923. This is a Laurent polynomial in t, can be computed directly from a knot diagram, and is (up to multiplication by some power of t) a knot invariant.

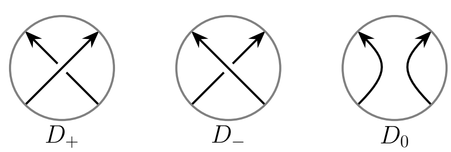

Figure 2.1: The diagrams involved in the skein relation

In the 1960's, Conway introduced a slightly modified version of the Alexander polynomial, which became known as the Alexander-Conway polynomial or the Conway polynomial $\nabla(z)$. Conway's reformulation and presentation of the polynomial emphasizes the *skein relation* : Let D_+ be a link diagram with a distinguished positive crossing, let D_- be the result of changing that crossing to be negative, and let

D_0 be the result of smoothing the crossing according to orientation (see Figure 2.1). Then the Conway polynomials of all three links satisfy:

$$\nabla(D_+) - \nabla(D_-) = z\nabla(D_0).$$

This formulations allows for a recursive method of calculating the knot invariant.

A little later, in the 1980's, Jones introduced the Jones polynomial $V(t)$, a Laurent polynomial (if the number of link components is odd, V_L is a Laurent polynomial over the integers, and if the number of link components is even, then V_L is $t^{1/2}$ times a Laurent polynomial) which also possesses a skein relation relating the Jones polynomials of links D_+, D_-, and D_0: [11]

$$t^{-1}V(D_+) - tV(D_-) = (t^{1/2} - t^{-1/2})V(D_0).$$

The HOMFLY(PT) polynomial was developed by four separate groups (Freyd-Yetter, Hoste, Lickorish-Millett, and Ocneanu) within a few months of each other in 1984, and the polynomial takes its name from the initials of its creators [6]. The later contributions of the pair Przytycki-Traczyk are often recognized with additional two letters added onto the name [15]. The HOMFLY(PT) polynomial can be formulated as a two-variable polynomial generalizes both the Conway polynomial and the Jones polynomial. As it was developed separately by four separate groups of mathematicians, each group has a different perspective and used different notational conventions. To emphasize the connection to the Jones and Conway polynomials, we will use the convention in which the polynomial is denoted $P(a, z)$ and its defining skein relation is:

$$a^{-1}P(D_+) - aP(D_-) = zP(D_0).$$

We observe that this specializes to the Jones and Conway polynomials by:

$$P(a=1,z)=\nabla(z) \qquad \text{and} \qquad P(a=t,z=t^{1/2}-t^{-1/2})=V(t),$$

and see that their defining skein relations are recovered.

Another great discovery of the 1980's, was the Kauffman state-sum model of the Jones polynomial.

Section 2.2

Kauffman State-Sum Model

Developed by Kauffman in 1987, the state-sum model is as follows: we begin with a link diagram and at every crossing, perform either an A-smoothing or a B-smoothing (see Figure 2.2)[9]. If the diagram had c crossings, then this results in 2^c possible

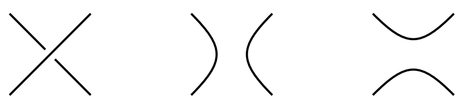

Figure 2.2: A crossing (left), its A-smoothing (middle), and B-smoothing (right)

arrangements of circles. Each such arrangement is called a *state*. Of particular interest to us are the A-state (where every crossing is given an A-smoothing) and the B-state (where every crossing is given a B-smoothing). On each state, we compute the *bracket*. Let σ denote a state. Then the bracket is defined by:

$$\langle D|\sigma\rangle := A^{\#A-\#B},$$

where $\#A$ is the number of A-smoothings in the state and $\#B$ is the number of B-smoothings in the state. The Kauffman bracket of a diagram D is the sum over all states σ of the bracket:

$$\langle D \rangle := \sum_{\sigma} \langle D|\sigma \rangle (-A^2 - A^{-2})^{|\sigma|-1},$$

where $|\sigma|$ is the number of circles in the state σ, and the Kauffman bracket polynomial is defined as follows:

$$\begin{aligned} F(D) &:= (-A)^{-3w(D)} \langle D \rangle \\ &= (-A)^{-3w(D)} \sum_{\sigma} \langle D|\sigma \rangle \\ &= (-A)^{-3w(D)} \sum_{\sigma} A^{\#A - \#B} (-A^2 - A^{-2})^{|\sigma|-1}. \end{aligned}$$

The Kauffman bracket $\langle D \rangle$ is not an link invariant; the value changes depending the choice of diagram. (It is invariant under Reidemeister moves II and III, but fails to be invariant under move I.)[9] The Kauffman bracket *polynomial*, however, is a link invariant. And in fact, with just a simple substitution, we obtain the Jones polynomial:

$$F(t^{-1/4}) = V(t).$$

This state-sum model allows to easily observe how changing a single crossing has a ripple effect through the Jones polynomial.

This state-sum model allows to easily observe how changing a single crossing has a ripple effect through the Jones polynomial. We can observe that the all-A-state will contribute a term of highest possible degree to the Kauffman bracket polynomial F, which means that, via the substitution $t^{-1/4}$ for A, this corresponds to the lowest possible degree of the Jones polynomial. That is,

$$\max\deg F \leq -3w(D) + c(D) + 2(A_D - 1),$$

(where A_D is the number of A-circles in the all-A state) and so with the replacement of $t^{-1/4}$ for A, we get a natural upper bound on the minimum degree of the Jones polynomial:

$$\min\deg V_D \geq \frac{-3w(D) + c(D) + 2(A_D - 1)}{-4} = \frac{3w(D) - c(D) - 2(A_D - 1)}{4}.$$

It can be helpful to rewrite this in terms of the number of negative crossings in the diagram. Let q denote the number of negative crossings in D. Then $w(D) = c(D) - 2q$, and we can write:

$$\begin{aligned}\min\deg V_D &\geq \frac{3(c(D) - 2q) - c(D) - 2(A_D - 1)}{4} \\ &= \frac{c(D) - A_D + 1}{2} - \frac{3q}{2}.\end{aligned}$$

By the same logic, the all B-state contributes a term of lowest possible degree in the Kauffman bracket polynomial, and thus of highest possible degree in the Jones polynomial. Thus

$$\max\deg V_D \leq \frac{-3w(D) - c(D) - 2(B_D - 1)}{-4} = \frac{3w(D) + c(D) + 2(B_D - 1)}{4} \qquad (2.2.1)$$

Incorporating information about negative crossings:

$$\max\deg V_D \leq \frac{3(c(D) - 2q) + c(D) + 2(B_D - 1)}{4} = c(D) + \frac{B_D - 1}{2} - \frac{3q}{2}. \qquad (2.2.2)$$

If D is A-adequate, then

$$\min\deg V_D = \frac{c(D) - A_D + 1}{2} - \frac{3q}{2}.$$

If D is B-adequate, then

$$\max\deg V_D = c(D) + \frac{B_D - 1}{2} - \frac{3q}{2}. \tag{2.2.3}$$

Further, we see that if D is positive, then it is A-adequate and has $q = 0$, and

$$\min\deg V_D = \frac{c(D) - A_D + 1}{2} \tag{2.2.4}$$

and

$$\max\deg V_D \leq c(D) + \frac{B_D - 1}{2}. \tag{2.2.5}$$

Chapter 3

The First Obstruction

Section 3.1

Bound on max degree of the Jones polynomial of a positive knot

Throughout the following, for a link diagram D we let:

- $c(D)$ be the crossing number
- $w(D)$ be the writhe
- $n(D)$ be the number of link components
- $s(D)$ be the number of Seifert circles
- $g(D)$ be the Seifert genus, which for a non-split link is found by $g(D) = \frac{c(D)-s(D)+2-n(D)}{2}$,
- $|A(D)|$ be the number of A-circles (the number of circles in the all-A state) (sometimes we may just use $|A|$ if there is only one diagram in question)
- $|B(D)|$ (or just $|B|$) be the number of B-circles

- V be the Jones polynomial
- V_i be the ith coefficient of the Jones polynomial.

  This means that if d is the lowest degree of a non-zero term in V, we write V_0 to refer to the coefficient of the term t^d, and then V_i is the (not necessarily non-zero) coefficient of the term t^{d+i}.

Whenever we refer to "an arc" of a diagram, we mean a portion of a strand that goes between two crossings, so an arc ends when it reaches any crossing, not just an undercrossing.

Definition 3.1.1. Let D be a link diagram. Smooth every crossing in the A-direction (see Figure 2.2). Create the dual graph, where each A-circle corresponds to a vertex in the dual graph, and each crossing shared between A-circles corresponds to an edge between vertices in the graph. This graph is called the **A-state graph**. (Figure 3.1)

Definition 3.1.2. Let D be a link diagram. The **reduced A-state graph** is the result of removing duplicate edges from the A-state graph, so that vertices in the reduced A-state graph share a single edge if and only if their corresponding A-circles share at least one crossing in the knot diagram.

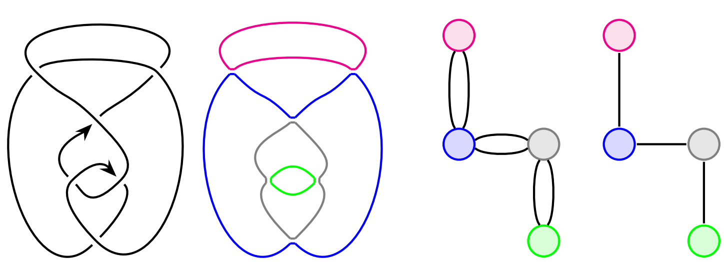

Figure 3.1: (Left to right:) A positive link diagram, its A-circles, its A-state graph, and its reduced A-state graph

Definition 3.1.3. A(n oriented) link diagram is called **positive** if every crossing in that diagram is positive. (See Figure 3.2.) A link is a **positive link** if it has a positive diagram.

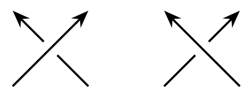

Figure 3.2: A positive crossing (left) and a negative crossing (right)

In everything that follows, we always assume we are working with non-split links.

Remark 3.1.4. In a positive diagram, A-smoothings are equivalent to smoothing according to Seifert's algorithm. So in a positive diagram, $|A(D)| = s(D)$.

Theorem 3.1.5. *(Stoimenow)[16]*

Let L be a positive link with positive diagram D. Then coefficient V_1 of the Jones polynomial satisfies:

$$(-1)^{n(L)-1}V_1 = s(D) - 1 - \#(\textit{pairs of Seifert circles that share at least one crossing}).$$

This means that $V_1 = 0$ exactly when the reduced A-state graph is a tree.

So if $V_1 = 0$ for a positive diagram, then there are exactly $s(D) - 1$ pairs of Seifert circles in the diagram that share at least one crossing. (See Figure 3.1.) For more details on A-state graphs, the Jones polynomial and its coefficients, and properties of positive knots (and almost-positive knots), see [16] and [17].

3.1.1. Positive link diagram D with $V_1 = 0$ must have $4 \min \deg V \geq c(D)$

We begin with theorems from knot theory folklore:

Theorem 3.1.6. *If L is a positive link of n components with positive diagram D, then*

$$\min\deg V_L = \frac{c(D) - s(D) + 1}{2}.$$

Lemma 3.1.7. *Let D be a reduced positive link diagram (i.e. with no nugatory crossings) such that $V_1 = 0$. Then $4 \min \deg V_L \geq c(D)$.*

Proof. Since D is positive and $V_1 = 0$, by Theorem 3.1.5 we know that the reduced A-state graph is a tree.

Consider a pair of adjacent vertices in the tree. If the corresponding vertices were connected by only one edge in the (unreduced) A-state graph, then the diagram would not be reduced. Since our diagram D is reduced, we conclude that any pair of adjacent vertices in the reduced A-state graph are connected by at least two edges in the A-state graph. Therefore the number of crossings in D is at least $2(s(D) - 1)$.

Observe that by Theorem 3.1.6,

$$\frac{c(D) - s(D) + 1}{2} = \min \deg V_L$$
$$2c(D) - 2(s(D) - 1) = 4 \min \deg V_L$$
$$c(D) = 2(s(D) - 1) + 4 \min \deg V_L - c(D)$$

As $c(D) \geq 2(s(D) - 1)$, we must have that $4 \min \deg V_L - c(D) \geq 0$ and so $4 \min \deg V_L \geq c(D)$.

□

Corollary 3.1.8. *Let L be a positive link with $V_1 = 0$. Then L has a positive diagram D for which $4 \min \deg V_L \geq c(D)$.*

3.1.2. Balanced and Burdened link diagrams

Proposition 3.1.9. *Let D_0 be a link diagram.*

(i) *Let D_1 be a diagram obtained from D_0 by smoothing a crossing according to Seifert's algorithm. Then*

 (i) $|n(D_1) - n(D_0)| = 1$, *and*

(ii) $||B(D_1)| - |B(D_0)|| = \begin{cases} 0 & \textit{if a negative crossing is smoothed, and} \\ 1 & \textit{if a positive crossing is smoothed.} \end{cases}$

(ii) *Let D_0 be a link diagram, and $(x_1, \ldots, x_m)$ a sequence of distinct crossings in D_0. Let D_i be the diagram obtained by smoothing (according to Seifert's algorithm) crossings x_1 through x_i. Then*

(i) $|n(D_i) - n(D_0)| \leq i$*, and*

(ii) $||B(D_i)| - |B(D_0)|| \leq i$*.*

Proof. Part $(i)(a)$ is clear: smoothing a crossing involving one component creates a new component, and smoothing a crossing involving two components combines them into one. For $(i)(b)$ - B-smoothing (as seen in Figure 2.2) a negative crossing does not change the total number of B-circles involved, but B-smoothing a positive crossing will either increase or decrease the number of B-circles by 1.

For part (ii), we rewrite $|n(D_i) - n(D_0)|$ as a telescoping sum to obtain the inequality

$$\begin{aligned} |n(D_i) - n(D_0)| &= |n(D_i) - n(D_{i-1}) + n(D_{i-1}) - n(D_{i-2}) + \cdots + n(D_1) - n(D_0)| \\ &\leq \sum_{k=0}^{i-1} |n(D_{k+1}) - n(D_k)|, \quad \text{which by Part } (a) \text{ is} \\ &= \sum_{k=0}^{i-1} 1 \\ &= i. \end{aligned}$$

Parallel argument for $|B|$. □

We now introduce the concepts of a balanced link diagram and a burdened link diagram.

Definition 3.1.10. A **balanced link diagram** is a (non-split) positive link diagram in which every pair of A-circles share exactly 0 or 2 crossings, and the reduced A-state graph is a tree.

An example appears in Figure 3.1. Observe that if $s(D)$ is the number of Seifert circles, then in a balanced link diagram D we have that $c(D) = 2(|A(D)| - 1) = 2(s(D) - 1)$.

Theorem 3.1.11. *Let D be a balanced link diagram. Then $n(D) = |B(D)|$. (The number of components of D is equal to the number of B-circles.)*

This is the key theorem needed to prove our main result, and the last section of this chapter leads up to a proof of Theorem 3.1.11.

Definition 3.1.12. A **burdened link diagram** is a (non-split) positive link diagram in which every pair of A-circles share 0 or at least 2 crossings, and the reduced A-state graph is a tree.

Observe that for all burdened link diagrams, there exists a (not necessarily unique, possibly empty) sequence of crossings $(x_1, \ldots, x_m)$ such that after Seifert-smoothing every x_i, we are left with a balanced link diagram. We call such a sequence a **Balancing sequence**.

Lemma 3.1.13. *Let D_0 be a burdened link diagram, and $(x_1, \ldots, x_m)$ a Balancing sequence for D_0. Then*

$$|B(D_0)| \leq 2m + n(D_0).$$

Proof. Proposition 3.1.9 tells us that $n(D_m) \leq m + n(D_0)$ and $|B(D_0)| \leq m + |B(D_m)|$. By definition of Balancing sequences, diagram D_m is balanced, and so $n(D_m) = |B(D_m)|$ by Theorem 3.1.11. Thus

$$|B(D_0)| \leq m + |B(D_m)| = m + n(D_m) \leq m + (m + n(D_0)) = 2m + n(D_0).$$

□

Corollary 3.1.14. *Let D_0 be a burdened link diagram of a link L. Then*

$$|B(D_0)| \leq 8 \min \deg V_L - 2c(D_0) + n(D_0).$$

Proof. Let $(x_1, \ldots, x_m)$ be a Balancing sequence, and D_m the balanced diagram obtained by smoothing all crossings in the sequence. Observe that then $s(D_m) = |A(D_m)| = |A(D_0)| = s(D_0)$, and consider the following:

$$\begin{aligned} c(D_0) - m &= c(D_m) && \text{and since } D_m \text{ is balanced, this is} \\ &= 2(s(D_m) - 1), \\ &= 2(s(D_0) - 1). && \text{Rearrange to obtain} \\ m &= c(D_0) - 2(s(D_0) - 1) \\ &= 2(c(D_0) - s(D_0) + 1) - c(D_0) \\ &= 4 \min \deg V(D_0) - c(D_0), && \text{by Theorem 3.1.6.} \end{aligned}$$

Then by Lemma 3.1.13,

$$|B(D_0)| \leq 2m + n(D_0) = 8 \min \deg V(D_0) - 2c(D_0) + n(D_0).$$

□

Theorem 3.1.15. *Let D be a burdened link diagram of a link L. Then*

$$\max\deg V_L \leq \frac{8\min\deg V_L + n(D) - 1}{2},$$

and when D is a knot diagram,

$$\max\deg V_K \leq 4\min\deg V_K.$$

Proof. Consider that the lowest possible degree term in the Kauffman bracket polynomial would be contributed by the all-B state. (For more details of the Kauffman bracket polynomial, we direct the reader to our introduction and to [9].) Since D is positive, the all-B state contribution to the Kauffman bracket polynomial is of degree $-3w - c - 2(|B| - 1) = -4c - 2|B| + 2$. Since no state can contribute a term of degree strictly less than that of the all-B state, we know that the minimal degree of the Kauffman bracket polynomial is greater than or equal to $-4c - 2|B| + 2$. This means that for the Jones polynomial V, we have

$$\begin{aligned}\max\deg V &\leq (-1/4)(-4c - 2|B| + 2)\\ &= \frac{2c + |B| - 1}{2}\\ &\leq \frac{2c + (8\min\deg V - 2c + n) - 1}{2} \quad \text{(by Cor. 3.1.14)}\\ &= \frac{8\min\deg V + n - 1}{2}.\end{aligned}$$

And if D is a knot diagram, then $n(D) = 1$ and so $\max\deg V \leq 4\min\deg V$. □

To remove the trouble of having to procure a burdened diagram, we can leverage known results about positive fibered links. In [16], as a corollary to 3.1.5, we have:

Corollary 3.1.16. *(Stoimenow) [16] For a positive link, $V_1 = 0$ if and only if L is fibered.*

(For more information about this theorem, about what it means for a link to be fibered, and for an overview of work done in these areas in particular having to do with state diagrams and knot positivity, we refer the reader to Stoimenow's work, and also to [8] and [7].)

This means we could have given an equivalent definition: a **burdened diagram** is a (non-split) reduced, positive link diagram of a fibered positive link.

With this observation, Theorem 3.1.15 gives us our main result:

Corollary 3.1.17. *If K is a fibered positive knot, then* $\max\deg V \leq 4 \min\deg V$.

Proof. Let D be a positive diagram of K. Since K is fibered, $V_1 = 0$ by 3.1.16. We can assume D is reduced, so we can assume that it is a burdened diagram, and then $\max\deg V \leq 4\min\deg V$ by Theorem 3.1.15. □

3.1.3. Completing the positivity classification of knots with crossing number ≤ 12

Positivity is already known for all but seven knots with crossing number ≤ 12. These remaining knots are $12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528}$, and 12_{n660}. We now have the tools to show that all of these knots are not positive.

"Positivity unknown" for these knots means that it was unknown whether the knot *or its mirror* is positive. As we have seen, it is a standard result that the minimum degree of the Jones polynomial of a positive knot is positive. Looking at the Jones polynomials given on KnotInfo [12] for our seven knots, we see that every exponent is negative. Therefore, the knots given are definitely not positive, but it remains to show that their mirrors could not have positive diagrams. The Jones polynomial of a knot's mirror is obtained by substituting t^{-1} for t, so the mirrors of our seven knots have Jones polynomials with all positive exponents. These polynomials are:

$$V(12_{n148}!) = t^3 + t^6 - 2t^7 + 3t^8 - 3t^9 + 3t^{10} - 3t^{11} + 2t^{12} - t^{13}$$
$$V(12_{n276}!) = t^3 + 2t^6 - 3t^7 + 4t^8 - 5t^9 + 4t^{10} - 4t^{11} + 3t^{12} - t^{13}$$
$$V(12_{n329}!) = t^3 + 2t^6 - 3t^7 + 3t^8 - 4t^9 + 4t^{10} - 3t^{11} + 2t^{12} - t^{13}$$
$$V(12_{n366}!) = t^3 - t^5 + 3t^6 - 4t^7 + 5t^8 - 5t^9 + 5t^{10} - 4t^{11} + 2t^{12} - t^{13}$$
$$V(12_{n402}!) = t^3 - t^7 + 2t^8 - t^9 + 2t^{10} - 2t^{11} + t^{12} - t^{13}$$
$$V(12_{n528}!) = t^3 - t^5 + 4t^6 - 5t^7 + 6t^8 - 7t^9 + 6t^{10} - 5t^{11} + 3t^{12} - t^{13}$$
$$V(12_{n660}!) = t^3 - 2t^5 + 5t^6 - 6t^7 + 8t^8 - 8t^9 + 7t^{10} - 6t^{11} + 3t^{12} - t^{13}$$

However, we also note that each knot is fibered, has $\min\deg V = 3$, and $\max\deg = 13$. Since $13 > 4(3) = 12$, our Corollary 3.1.17 tells us that in fact none of these seven knots can be a positive knot. This completes the classification of all knots of crossing number ≤ 12 as positive or not positive.

Section 3.2

Proof of Theorem 3.1.11

This section will culminate in a proof of our key theorem from the previous section:

Theorem 3.1.11 *Let D be a balanced link diagram. Then $n(D) = |B(D)|$.*

In this section we will always assume that D is a balanced diagram, and we will need to introduce some new definitions.

Definition 3.2.1. Let A and A' be two A-circles that share crossings. Since D is balanced, they share exactly two crossings. Call these crossings a **matching pair**.

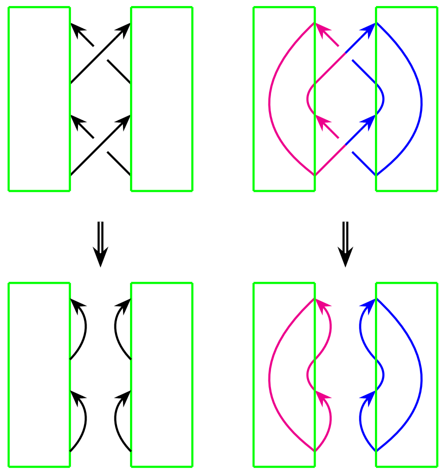

Figure 3.3: When we smooth a matching pair, we disconnect the diagram

Since the (underlying reduced A-graph of this) diagram has a tree structure, if we smooth a matching pair we will disconnect the diagram. (See Figure 3.3.)

Observe that for a balanced diagram D, we have a bijection between the set of matching pairs and the set of edges in the reduced A-state graph. Since this graph is a tree, if we start on an A-circle and pass through a matching pair (by following along a component or a B-circle), we can only return to the A-circle by first passing through that matching pair again.

Then a B-circle must pass through a matching pair an even number of times, and also B-circles cannot cross themselves, which leaves us with four possibilities of how a B-circle could run through a matching pair (see Figure 3.4).

Similarly, since components also straddle matching pairs and have a prescribed orientation, we are left with four possibilities of how a component could run through a matching pair (see Figure 3.5).

Definition 3.2.2. We say that a matching pair is **synchronized** if arcs of the matching pair that belong to the same A-circle are also part of the same B-circle if and

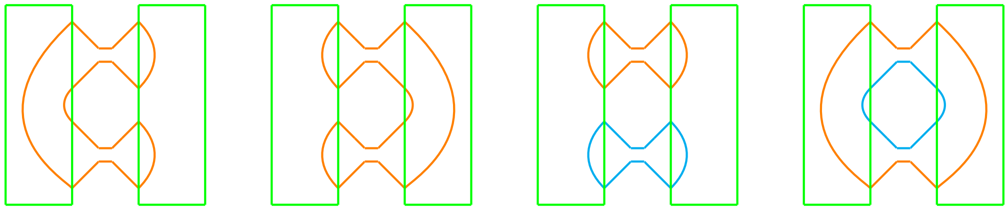

Figure 3.4: B-circle possibilities for a matching pair

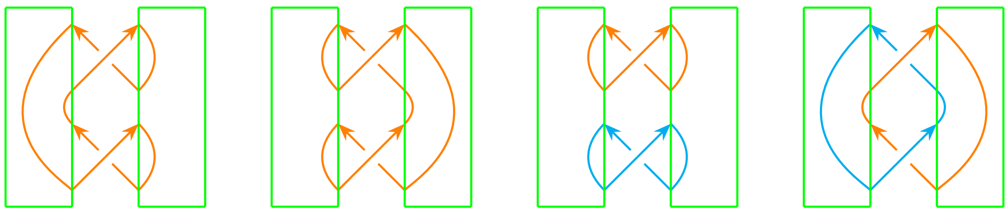

Figure 3.5: Component possibilities for a matching pair

only if they are part of the same component.

(So, for example, if a matching pair has the third possibility for B-circles shown in Figure 3.4, then it is a synchronized matching pair if and only if it has the third possibility for components shown in Figure 3.5.)

In Figure 3.6, we see an example of a balanced diagram in which *every* matching pair is synchronized. In this example, we can observe that for any arcs x and y that are part of the blue A-circle, those two arcs are part of the same B-circle if and only if they are part of the same component. It turns out that this will always be the case when we have a balanced diagram in which every matching pair is synchronized. While reading the proof of this, it may be helpful to look back at the diagram in Figure 3.6.

Definition 3.2.3. We say that a balanced link diagram is **synchronized** if arcs on the same A-circle are also part of the same B-circle if and only if they are part of the same component

Lemma 3.2.4. *A balanced diagram is synchronized if and only if all of its matching pairs are synchronized.*

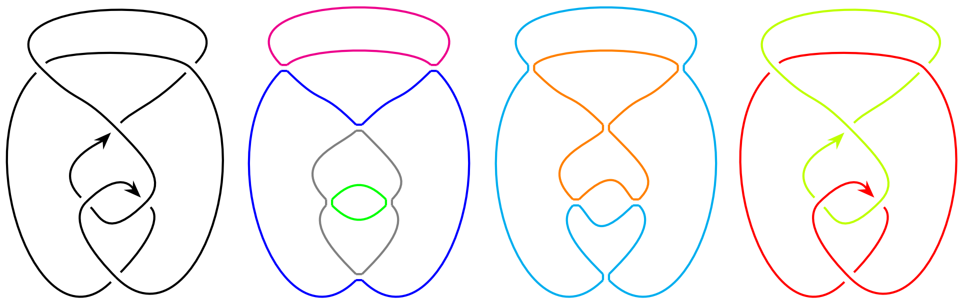

Figure 3.6: (Left to right:) A balanced diagram, its A-circles, its B-circles, and its components

Proof. If a balanced diagram is synchronized, then automatically all of its matching pairs are synchronized.

Now we would like to show that if all matching pairs are synchronized, then the balanced diagram must also be synchronized. Let x and y be distinct arcs on A-circle A that are part of the same component C. We would like to show that x and y must then also be part of the same B-circle.

We travel along C from x to get to y, and since these arcs are distinct, we must pass through matching pairs on our way. When we pass through a matching pair, we change A-circles. Since the underlying reduced A-state graph is a tree in which each edge uniquely corresponds to a matching pair of crossings in our diagram D, we must pass through the same matching pair again in order to return to A. Since every matching pair is synchronized, when we return to A we do so along an arc that is part of the same B-circle as x, by definition of a synchronized matching pair. Since this is true every time we leave A, when we arrive at arc y we see that it also must be part of the same B-circle as arc x.

Thus arcs on the same A-circle that are part of the same component must also be part of the same B-circle.

By swapping the roles of component and B-circle in the paragraph above we get

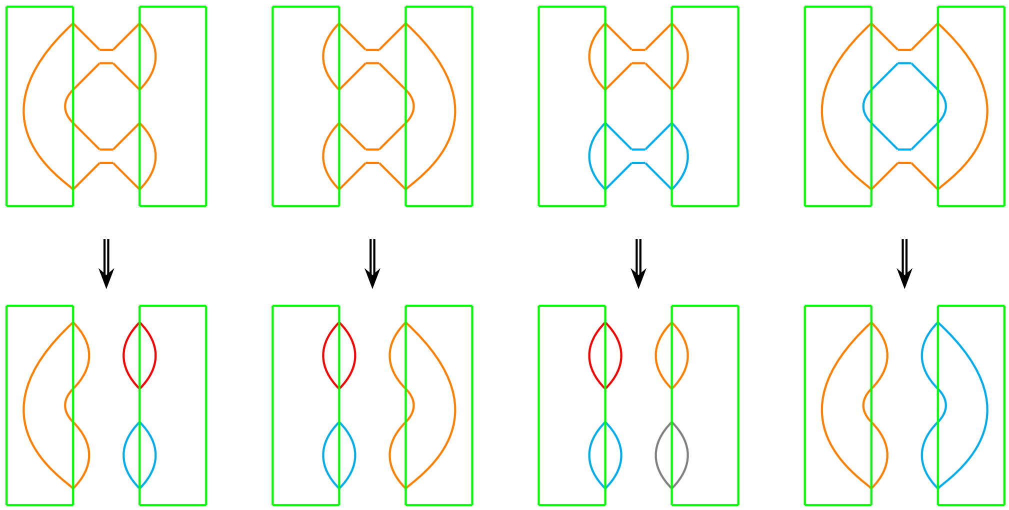

Figure 3.7: **Top row:** B-circle possibilities for a matching pair; **Bottom row:** B-circles in the two new smaller diagrams that result from smoothing the matching pair

the reverse implication, and so we conclude that arcs on the same A-circle are part of the same B-circle if and only if they are part of the same component, and thus the diagram is synchronized by definition.

□

Lemma 3.2.5. *In a balanced diagram every matching pair is synchronized.*

Proof. We proceed by induction on $|A|$, the number of A-circles. Our claim clearly holds in the base case $|A| = 1$.

Now suppose our claim is true for any diagram with $|A| = k$ for some $k \in \mathbb{N}$. Consider a diagram D for which $|A(D)| = k + 1$. Choose any matching pair and Seifert smooth it. Because of the underlying tree structure, this disconnects our diagram and leaves us with two smaller diagrams. Call them D' and D''. This is pictured in Figures 3.7 and 3.8.

Observe that D' and D'' are balanced link diagrams themselves, and each has strictly fewer A-circles than our original D. So by the induction hypothesis, each

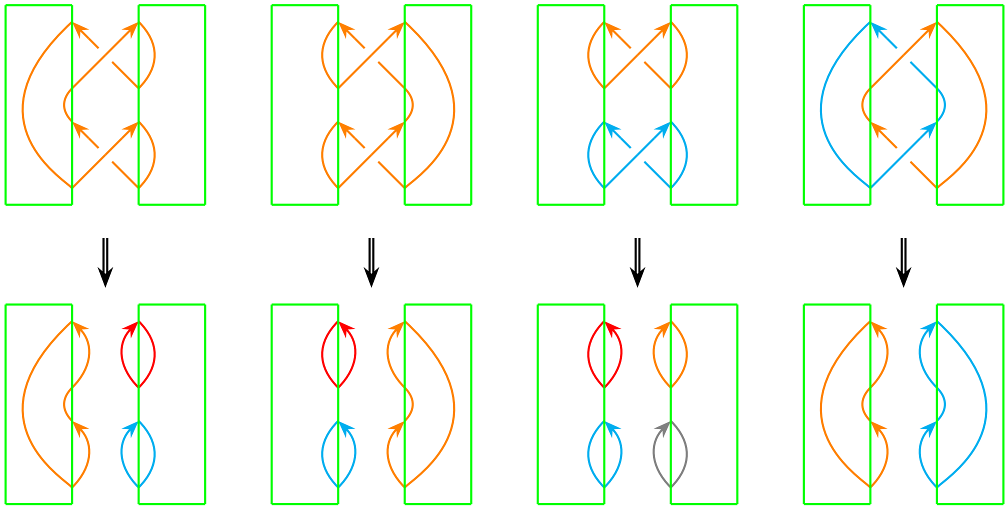

Figure 3.8: **Top row:** Component possibilities for a matching pair; **Bottom row:** Components in the two new smaller diagrams that result from smoothing the matching pair

matching pair in D' (and in D'') is synchronized. Then by our previous Lemma 3.2.4, we have that D' and D'' are both synchronized.

By looking at the options of combinations in Figures 3.7 and 3.8, we see that then the diagrams D' and D'' could only have come into existence by smoothing a synchronized matching pair in D. But since we chose an arbitrary matching pair in D to smooth, this means that every matching pair in our larger diagram D is a synchronized matching pair. □

Theorem 3.2.6. *Balanced diagrams are synchronized.*

Proof. Follows from Lemma 3.2.5 and Lemma 3.2.4 □

Now at last we are ready for Theorem 3.1.11:

Theorem 3.1.11. *Let D be a balanced link diagram. Then $n(D) = |B(D)|$. (The number of components is equal to the number of B-circles.)*

Proof of Theorem 3.1.11. Let D be a balanced diagram. Then the reduced A-state graph of D is a tree, and therefore has a leaf. Since the set of matching pairs of D is in bijection with the set of crossings in the reduced A-state graph, this means that D contains a "leaf A-circle" - an A-circle that is only incident to a single matching pair.

Therefore, just as a tree of $k+1$ vertices can be viewed as the result of attaching a single vertex to a tree of k vertices, so too can a balanced diagram D where $|A(D)| = k+1$ be viewed as the result of attaching a single A-circle to a balanced diagram D' with $|A(D')| = k$.

Now we proceed by induction on the number of A-circles. Observe that in the base case of $|A| = 1$, the result holds. Let D be a balanced diagram with $|A(D)| = k+1$. Then there exists some balanced diagram D' such that $|A(D')| = k$, and an A-circle A' in D' such that D can be obtained by grafting a leaf onto D' along two (not necessarily distinct) arcs x and y of A'.

By inductive hypothesis, the claim is true for D' so $|B(D')| = n(D')$.

Consider those arcs x and y in D'. If they are part of the same link component, then we create a new component by attaching a leaf circle. If instead x and y are part of different link components, then attaching a leaf combines these two components into one. Similarly, if x and y are part of the same B-circle, then we create a new B-circle by attaching a leaf, and if x, y are part of different B-circles, then attaching a leaf combines them into one.

On the surface, we seem to be looking at four different possibilities of changes in number of components and number of B-circles that could result from attaching a leaf circle. However, we know from Theorem 3.2.6 that the diagram is synchronized, which means that arcs on the same A-circle are from the same B-circle if and only if they are from the same component. This leaves us with exactly two possibilities: that adding a leaf creates a new component and a new B-circle, or adding a leaf combines two components and combines two B-circles.

By our inductive hypothesis, we know that D' has the same number of B-circles as components. So when we add a leaf to obtain D we either increase both by 1 or we decrease both by 1 – but in either case, the number of B-circles will still equal the number of components in our new diagram D.

So by induction, in any balanced diagram the number of link components is equal to the number of B-circles.

□

Chapter 4

The Second and Third Obstructions

In this chapter we find a similar result for the case of positive links with second Jones coefficient equal to ± 1 or ± 2. We focus on two kinds of positive link diagrams that we call balanced and oddly balanced diagrams of type 1 and type 2, generalizing the definitions presented in the last chapter. In Theorems 4.3.10 and 4.3.11, we prove that in a balanced diagram, the number of B-circles is equal to the number of link components. In Theorem 4.3.12, we prove that in an oddly balanced diagram, the number of B-circles either is equal to the number of link components, or the two quantities differ by ± 2. As such, for balanced (and oddly balanced) diagrams we are able to replace the diagram-dependent quantity B_D in 2.2.5 with a diagram-independent quantity.

Burdened diagrams are the positive diagrams which can be sanded down into balanced diagrams by smoothing m crossings, for some non-negative integer m. This means that, as we saw in the case of fibered positive links, our bound 2.2.5 can be rewritten in terms of this m.

By delving deeper into the structural properties of balanced and burdened diagrams, we are able to replace all of those diagram dependent quantities (crossing number, number of B-circles and smoothing number m) with diagram independent quantities (minimal degree of the Jones polynomial, number of link components, and leading coefficient of the Conway polynomial).

We are able to prove that:

Theorem. 4.2.17 *Let L be a positive link with n link components, Jones polynomial V_L, and Conway polynomial ∇_L.*

(a) *If the second coefficient of V_L is ± 1, then*

$$\max\deg V_L \leq 4\min\deg V_L + \frac{n-1}{2} + 2\,\text{lead coeff}\,\nabla_L - 2.$$

(b) *If the second coefficient of V_L is ± 2, then*

$$\max\deg V_L \leq 4\min\deg V_L + \frac{n-1}{2} + \text{lead coeff}\,\nabla_L.$$

In Section 4.6, we present two examples of infinite families of knots, and use these two theorems to prove that these knots cannot be positive, and are instead almost-positive. We also present a related infinite family of knots whose non-positivity can be shown using the result of Theorem 3.1.15

Section 4.1

Background

In everything that follows, we always assume we are working with non-split links, and for a link diagram D we will use the following change in notation.

- A_D is the number of A-circles (sometimes we may just use A if there is only one diagram in question)

- B_D (or just B) is the number of B-circles
- ∇ is the Conway polynomial, and ∇_D (or $\nabla(D)$, ∇_L, or $\nabla(L)$) is the Conway polynomial of link L represented by D.
- V is the Jones polynomial, and V_D (or $V(D)$, V_L, or $V(L)$) is the Jones polynomial of the link L represented by diagram D

Definition 4.1.1. The ***second coefficient of the Jones polynomial*** is the coefficient of the term with exponent one higher than the minimal degree. For example, the Jones polynomial of the trefoil is $V(3_1) = t + t^3 - t^4$, so we have $\min\deg V = 1$, $\max\deg V = 4$, and the ***second Jones coefficient*** is equal to 0 in this case.

Section 4.2

Balanced and Burdened Diagrams

4.2.1. Balanced Diagrams

Examples of each of the following appear in Figures 4.1-4.4.

Definition 4.2.1. A **balanced diagram of type 0** is a (non-split) positive link diagram D for which:

(a) The reduced A-state graph of D is a tree, and

(b) Every pair of A-circles share exactly 0 or 2 crossings.

Definition 4.2.2. A **hole** in a graph is an interior face.

Definition 4.2.3. A **balanced diagram of type 1** is a (non-split) positive link diagram D for which:

(a) The reduced A-state graph of D (denoted G'_D) has exactly 1 hole (interior face), and

(b) For each pair v, w of A-circles, exactly one of the following is true:

(i) v and w share 0 crossings,

(ii) v and w share exactly 1 crossing, and the edge in G'_D corresponding to that crossing is part of a cycle, or

(iii) v and w share exactly 2 crossings, and the edge in G'_D corresponding to those crossings is not part of a cycle.

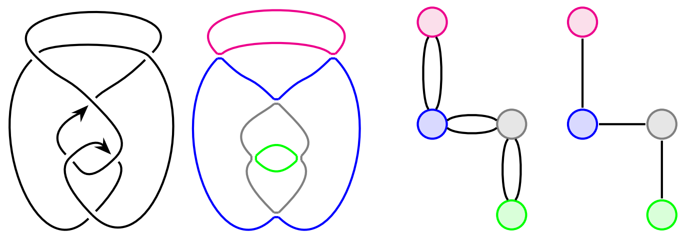

Figure 4.1: (Left to right:) A balanced diagram of type 0, its A-circles, its A-state graph, and its reduced A-state graph

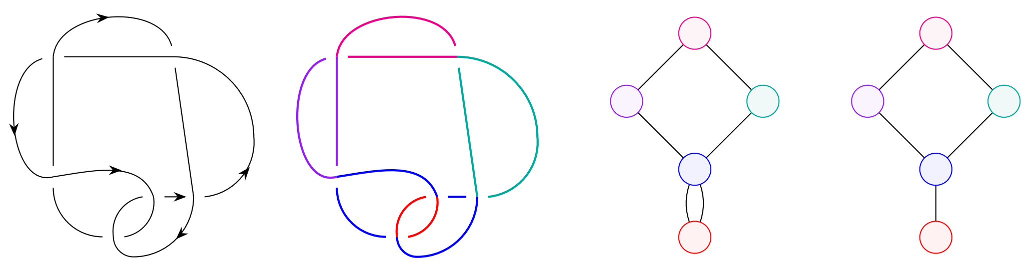

Figure 4.2: A balanced diagram of type 1, its A-circles, its A-state graph, and its reduced A-state graph

Definition 4.2.4. A **balanced diagram of type 2** is a (non-split) positive link diagram D for which:

(a) The reduced A-state graph of D (denoted G'_D) has exactly 2 holes, and

(b) For each pair v, w of A-circles, exactly one of the following is true:

(i) v and w share 0 crossings,

(ii) v and w share exactly 1 crossing, and the edge in G'_D corresponding to that crossing is part of a cycle, or

(iii) v and w share exactly 2 crossings, and the edge in G'_D corresponding to those crossings is not part of a cycle.

(c) An even number of edges in G'_D are part of cycles.

Definition 4.2.5. If D is a positive link diagram that satisfies criteria (1) and (2) in Definition 4.2.4 but an odd number of edges in its reduced A-state graph are part of cycles, then we call D an **oddly balanced diagram of type 2**.

Definition 4.2.6. A **k-balanced diagram** is a balanced diagram of type 1 whose one hole (interior face) in its reduced A-state graph is bounded by k edges (where we ignore any non-cycle edges that might "protrude" into the hole).

A **$(\mathbf{k_1}, \mathbf{k_2})$-balanced (or oddly balanced) diagram** is a balanced (or oddly balanced) diagram of type 2 whose reduced A-state graph contains 2 holes, with one hole bounded by k_1 edges (again ignoring any non-cycle edges) and the other bounded by k_2 edges (these two sets of edges are not necessarily disjoint).

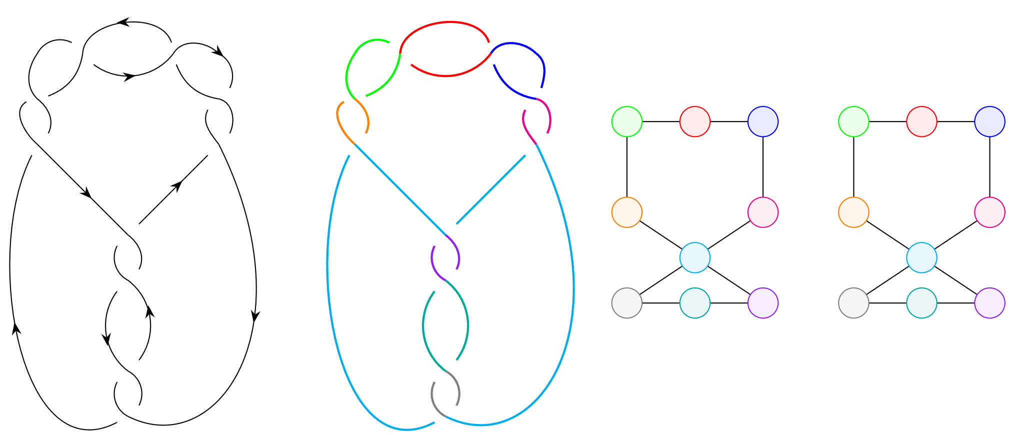

Figure 4.3: A balanced diagram of type 2, its A-circles, its A-state graph, and its reduced A-state graph

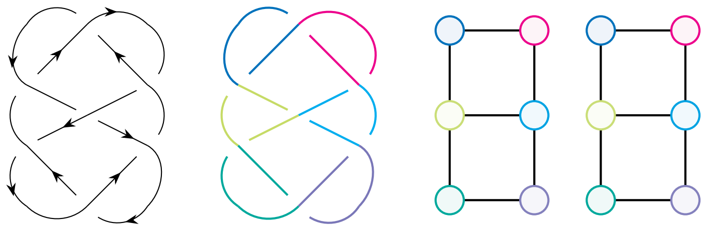

Figure 4.4: An oddly balanced diagram of type 2, its A-circles, A-state graph, and its reduced A-state graph

4.2.2. Burdened Diagrams

To generalize to a larger class of positive link diagrams, we consider *burdened diagrams*. We picture burdened diagrams as would-be balanced diagrams that are burdened by extra crossings, making them violate criteria (b) or (c) in the above definitions. Examples of burdened diagrams appear in Figure 4.5 and Figure 4.6.

Definition 4.2.7. A **burdened diagram of type 0** is a (non-split) positive link diagram D for which:

(a) The reduced A-state graph of D (denoted G'_D) is a tree, and

(b) Every pair of A-circles share 0 or at least 2 crossings.

Definition 4.2.8. A **burdened diagram of type 1** is a (non-split) positive link diagram D for which:

(a) The reduced A-state graph of D (denoted G'_D) has exactly 1 hole (interior face), and

(b) For each pair v, w of A-circles, exactly one of the following is true:

(i) v and w share 0 crossings,

(ii) v and w share at least 1 crossing, and the edge in G'_D corresponding to that crossing is part of a cycle, or

(iii) v and w share <u>at least</u> 2 crossings, and the edge in G'_D corresponding to those crossings is not part of a cycle.

Definition 4.2.9. A **<u>burdened diagram of type 2</u>** is a (non-split) positive link diagram D for which:

(a) The reduced A-state graph of D (denoted G'_D) has exactly 2 holes, and

(b) For each pair v, w of A-circles, exactly one of the following is true:

(i) v and w share 0 crossings,

(ii) v and w share <u>at least</u> 1 crossing, and the edge in G'_D corresponding to that crossing is part of a cycle, or

(iii) v and w share <u>at least</u> 2 crossings, and the edge in G'_D corresponding to those crossings is not part of a cycle.

(c) An even number of edges in G'_D are part of cycles.

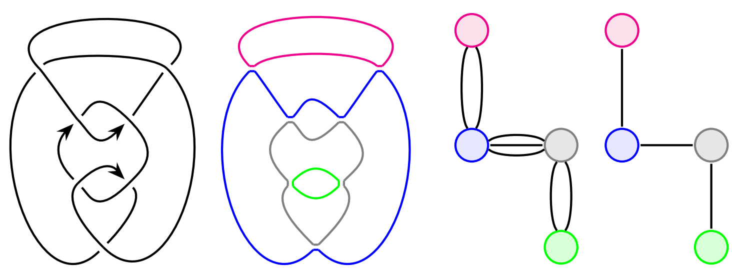

Figure 4.5: (Left to right:) A burdened diagram of type 0, its A-circles, its A-state graph, and its reduced A-state graph

Definition 4.2.10. If D is a positive link diagram that satisfies criteria (1) and (2) in Definition 4.2.9 but an odd number of edges in G'_D are part of cycles, then we call D an **<u>oddly burdened diagram of type 2</u>**.

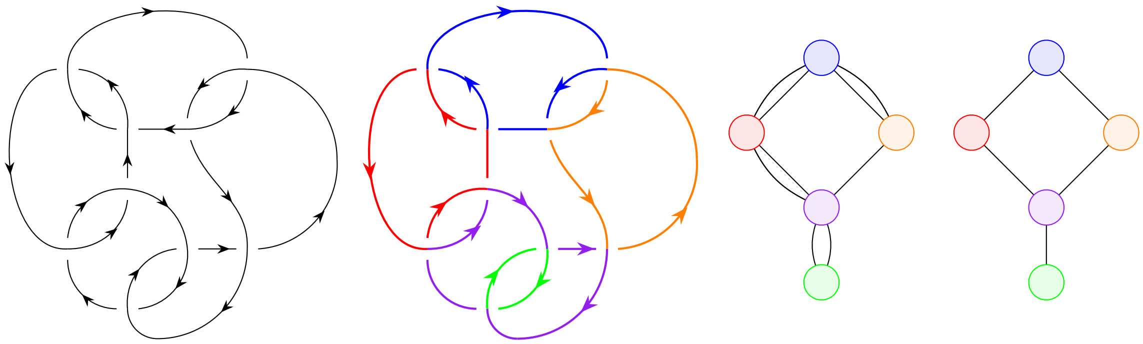

Figure 4.6: A burdened diagram of type 1, its A-circles, A-state graph, and reduced A-state graph

Definition 4.2.11. A **k-burdened diagram** is a burdened diagram of type 1 whose one hole in its reduced A-state graph is bounded by k edges (where we ignore any cut edges that might "protrude" into the hole).

A **$(\mathbf{k_1}, \mathbf{k_2})$-burdened (or oddly burdened) diagram** is a burdened (or oddly burdened) diagram of type 2 whose reduced A-state graph contains 2 holes, with one hole bounded by k_1 edges (again ignoring any cut edges) and the other bounded by k_2 edges (note that the two sets of edges are not necessarily disjoint).

Remark 4.2.12. For links with k-burdened diagrams of type 1, the value of k is unique. That is, if D is a k-burdened diagram and D' is a k'-burdened diagram for $k \neq k'$, then D and D' do not represent the same link. (We will see later on that this follows from Lemma 4.5.15, which says that this k will be twice the leading coefficient of the Conway polynomial of the link.) However, this is not true for diagrams of type 2. In Figure 4.7 we see an example of a $(6, 4)$-balanced diagram that is equivalent to an $(8, 4)$-oddly balanced diagram. Not only are the values of k_1 and k_2 not unique, the set of knots with balanced type 2 diagrams and the set of knots with oddly balanced type 2 diagrams are not mutually exclusive.

In this chapter, we develop the following bounds on the Jones polynomial of

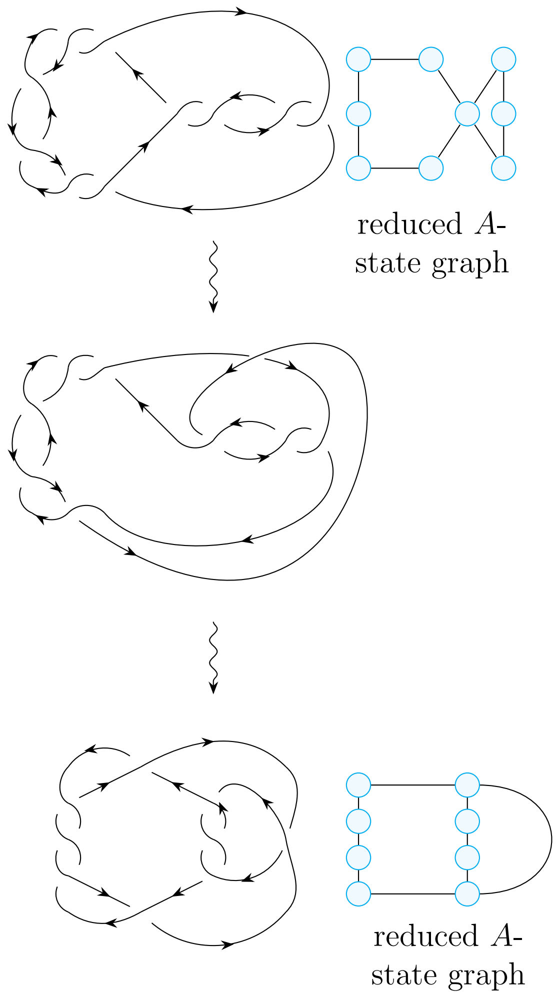

Figure 4.7: A $(6, 4)$-balanced diagram (top) that is equivalent to an $(8, 4)$-oddly balanced diagram (bottom)

burdened diagrams.

Theorem 4.2.13. *Let L be a link with a burdened diagram of type 1. Then*

$$\max\deg V_L \leq 4\min\deg V_L + \frac{n-1}{2} + 2\,\text{lead coeff}\,\nabla_L - 2,$$

where V is the Jones polynomial, ∇ is the Conway polynomial, and n is the number of link components.

Theorem 4.2.14. *Let L be a link with a burdened diagram of type 2 or an oddly burdened diagram of type 2. Then*

$$\max \deg V_L \leq 4 \min \deg V_L + \frac{n-1}{2} + \text{lead coeff } \nabla_L,$$

where V is the Jones polynomial, ∇ is the Conway polynomial, and n is the number of link components.

This allows to generalize to larger classes of positive links, because the second Jones coefficient tells us about the number of holes in the reduced A-state graph of a positive link diagram.

Theorem 4.2.15. *(Stoimenow)[16]*

Let L be a positive link with positive diagram D. Then the second coefficient V_1 of the Jones polynomial satisfies:

$$(-1)^{n(L)-1} V_1 = s(D) - 1 - \#(\textit{pairs of Seifert circles that share at least one crossing}).$$

As Stoimenow notes, this means the absolute value of second Jones coefficient is exactly the first Betti number of the reduced Seifert graph. Since in positive diagrams the Seifert circles are the A-circles, this means that the absolute value of second coefficient of the Jones polynomial does indeed count the number of holes in the reduced A-state graph of a positive diagram D. Since the Jones polynomial is a diagram-independent link invariant, this means that <u>every</u> positive diagram of a positive link with second Jones coefficient $0, \pm 1$, or ± 2 will have $0, 1$, or 2 (respectively) holes in its reduced A-state graph.

Corollary 4.2.16. (a) *Every reduced positive diagram of a link with second Jones coefficient equal to 0 is a burdened diagram of type 0.*

(b) *Every reduced positive diagram of a link with second Jones coefficient equal to* ± 1 *is a burdened diagram of type* 1*.*

(c) *Every reduced positive diagram of a link with second Jones coefficient equal to* ± 2 *is either a burdened diagram of type* 2 *or an oddly burdened diagram of type* 2*.*

Proof. Let L be a positive link with second Jones coefficient equal to $0, \pm 1$, or ± 2. Let D be a reduced positive link diagram of L. By 4.2.15, the reduced A-state graph of D must contain exactly $0, 1$, or 2 (respectively) holes. If any pair of A-circles v, w share exactly one crossing, then the corresponding edge in the reduced A-state graph is part of a cycle (otherwise, the diagram D is not reduced).

Thus D indeed satisfies definition 4.2.7, 4.2.8, 4.2.9, or 4.2.10. □

Theorem 4.2.17. *Let L be a positive link with n link components, Jones polynomial V_L, and Conway polynomial ∇_L.*

(a) *If the second coefficient of V_L is ± 1, then*

$$\max\deg V_L \leq 4\min\deg V_L + \frac{n-1}{2} + 2\,\text{lead coeff}\,\nabla_L - 2.$$

(b) *If the second coefficient of V_L is ± 2, then*

$$\max\deg V_L \leq 4\min\deg V_L + \frac{n-1}{2} + \text{lead coeff}\,\nabla_L,$$

where lead coeff *is the coefficient of the highest degree term.*

Proof. This follows directly from Corollary 4.2.16 and Theorems 4.2.13 and 4.2.14. □

This can be used as a positivity obstruction. In Section 4.6, we present infinite families of knots which can be shown not to be positive using these theorems.

Section 4.3

Clasping Positive Diagrams

In this section, we explore some properties of positive diagrams as we prepare to generalize the result of our last chapter:

Theorem 4.3.1. *[2] Let D be a balanced diagram of type 0 with n link components. Then*

$$B_D = n.$$

(The number of B-circles is equal to the number of link components. This justifies our use of the word balanced.)

Whenever we refer to "an arc" of a diagram, we mean a portion of a strand that goes between two crossings, so an arc ends when it reaches any crossing, not just an undercrossing.

Definition 4.3.2. In a connected graph, a **cut edge** is an edge whose deletion disconnects the graph. An edge is a cut edge if and only if it is not part of a cycle.

Proposition 4.3.3. *Every cycle in the reduced A-state graph of a positive diagram is even.*

Proof. It suffices to prove that this is true for any diagram D whose reduced A-state graph G contains no cut edges, since cut edges will never be part of a cycle.

Let (v_1, v_2) be an edge in G. Since G contains no cut edges, this edge is contained in some cycle $P = (v_1, v_2, \ldots, v_r, v_1)$. We know that there are no duplicate edges in G, so $r \geq 3$.

In the diagram D, A-circles A_1 and A_2 (corresponding to vertices v_1 and v_2) are connected by a crossing, and A_r and A_1 are also connected by a crossing. Either (without loss of generality) A_2 and A_r are both nested inside A_1, or none of the A_i's are nested inside any of the others. If two circles share a crossing and are not nested, then one has a clockwise orientation and the other has a counterclockwise orientation. If two circles share a crossing and one is nested inside the other, then they have the same orientation.

Assume A_1 has counter-clockwise orientation. We can note this with a "+" and form a sequence that indicates the orientations of each circle:

- if none of the circles are nested, we have (alternating) sequence

$$(A_1, A_2, A_3, \ldots, A_r, A_1) : (+, -, +, \ldots, -, +),$$

- or if the circles are nested inside A_1, we have (alternating if not for the contribution of A_1) sequence

$$(A_1, A_2, A_3, \ldots, A_r, A_1) : (+, +, -, \ldots, +, +).$$

In either case, notice that the sequence must have r even for the rest of the entries to alternate between + and −. □

Because of this fact, the reduced A-state graph of any positive diagram of any positive link with second Jones coefficient $\neq 0$ must contain a cycle of length ≥ 4. Meaning, the smallest possible balanced diagram of type 1 has at least four crossings.

In Figure 4.8 we see a balanced diagram of type 1 that we are able to transform into a balanced diagram of type 0 by adding a clasp. Let A_1, A_2, A_3, A_4 be the A-circles of this diagram, with corresponding vertices v_1, v_2, v_3, v_4 forming a cycle in the A-state graph. We take arcs in the diagram that belong to some A-circles A_1 and A_3, and interlock them. What happened in the A-state graph? All edges of the form

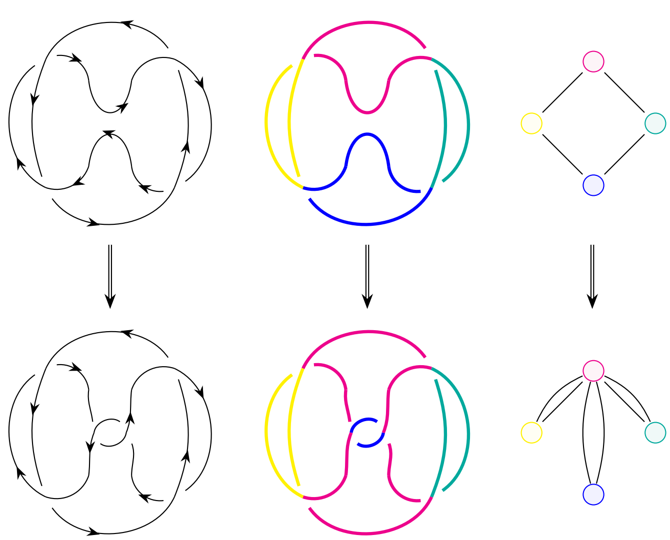

Figure 4.8: Clasping a 4-Balanced diagram of type 1 to obtain a Balanced diagram of type 0

(v_i, v_3) are now of the form (v_i, v_1), and we have added two copies of the new edge (v_1, v_3). This clasping did not change the number of A-circles in D (or the number of vertices of G), but it did change how the circles (and their corresponding edges in G) are arranged relative to one another. This is shown in a little more generality in Figure 4.9, which demonstrates how we can view this as one A-circle swallowing the other. This kind of clasping clearly does not change the number of link components. But more interestingly, we can see in Figure 4.10 that this does not change the number of B-circles either. Hopefully, by repeatedly forming these clasps, we can kill a hole in a reduced A-state graph and preserve the number of link components and the number of B-circles as we do so.

Since the A-state graph will tell us information to help us classify a diagram as balanced or not, we would like to be able to think about performing *clasp moves* directly from the information contained in the graph, and not have to have a diagram to start with.

4.3.1. The Clasp Move

Let D be a reduced positive link diagram with the following properties: Every cut edge in the reduced A-state graph of D corresponds to exactly two crossings in D, and every cycle edge in the reduced A-state graph corresponds to exactly one crossing in D.

Suppose there is a path (v_1, v_2, v_3) in the reduced A-state graph that is part of a cycle, and that v_2 has degree 2. Then there are arcs a_1 and a_3 in the diagram D (that are part of the A-circles corresponding to vertices v_1 and v_3) that can be clasped together with positive crossings. We saw this in Figure 4.8.

Now, suppose that: (1) (v_1, v_2, v_3) is part of a cycle in the reduced A-state graph, (2) Cutting edges (v_1, v_2) and (v_2, v_3) disconnects the graph, and (3) this disconnects it so that the component containing v_2 is a tree. So, this means that v_2 may not have degree 2 in the graph, but from the perspective of any vertex in the graph that is not part of that tree rooted at v_2, v_2 might as well have degree 2 – every path from v_2 to any other vertex (that is not part of the tree) must use edge (v_1, v_2) or (v_2, v_3). Then, as before, we can clasp together arcs in the link diagram corresponding to v_1 and v_3. This is shown in Figure 4.11, and will be called a *clasp move*.

If a reduced positive link diagram D satisfies the criteria listed above for performing a clasp move, then we say that D is *claspable*:

Definition 4.3.4. A link diagram D is **claspable** if it satisfies the following:

(a) D is a reduced positive link diagram

(b) Every cut edge in the reduced A-state graph of D corresponds to exactly two edges in the A-state graph of D (and thus also corresponds to exactly two crossings in D), and every cycle edge in the reduced A-state graph corresponds to exactly one edge in the A-state graph (and thus to exactly one crossing in

D).

(c) The reduced A-state graph contains a cycle $(v_1, v_2, v_3, \ldots, v_1)$ such that:

(i) Edges (v_1, v_2) and (v_2, v_3) form a cut set in the graph

(ii) Cutting those two edges disconnects the graph so that the component containing v_2 is a tree

Definition 4.3.5. We perform a **clasp move** on a claspable diagram D (and its associated A-state graph G) by clasping arcs a_1 and a_3 together with positive crossings (transferring all edges incident to v_3 to be incident to v_1, and then adding in two copies of the edge (v_1, v_3)).

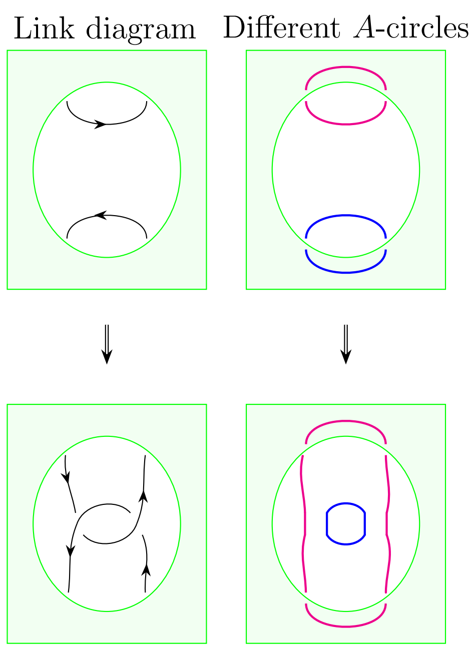

Figure 4.9: Adding a clasp with positive crossings

Proposition 4.3.6. *Let D be a claspable diagram, and let D' be the diagram obtained by performing a clasp move. Let G be the A-state graph of D, and let G' be that of D'. Then G and G' have the same vertex set. If some vertex v_i is incident to only one vertex in G, then v_i is still incident to only one vertex in G'.*

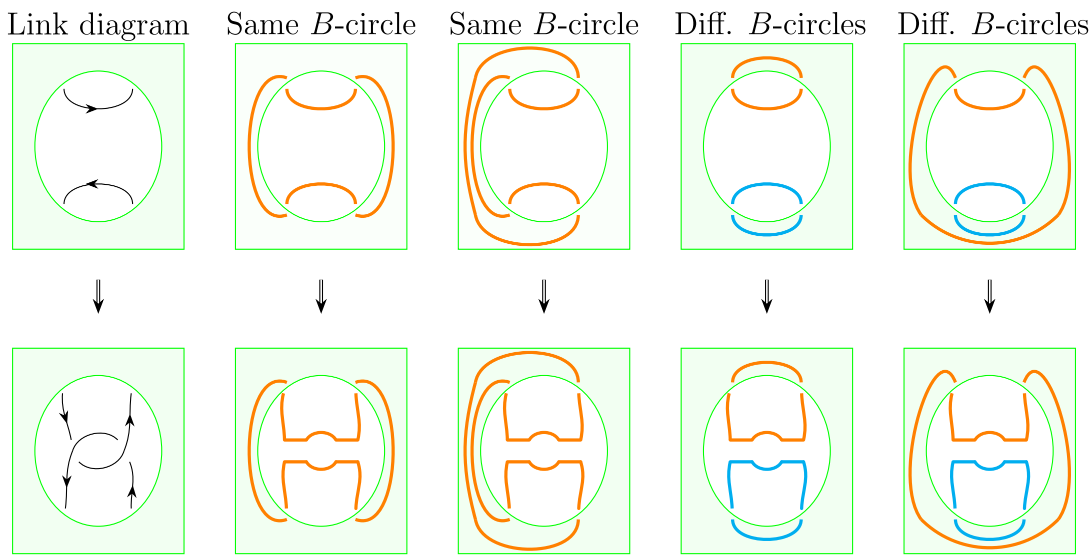

Figure 4.10: Adding a clasp with positive crossings does not change the number of B-circles.

Proof. Suppose for contradiction that v_i is incident to more than one vertex in G'. The clasp move takes all edges in the A-state graph G of the form (v_3, v_j) and makes them edges of the form (v_1, v_j) in G', and all other edges remain exactly the same.

So the only way a vertex could have a different set of neighbors in G' than in G is if the vertex itself were v_3, or if one of its neighbors were v_3. Since by assumption v_3 is part of a cycle in the reduced A-state graph and v_i is not, it must be the case that v_i was incident to v_3 in G. But then now v_i is only incident to v_1 in G'. □

Lemma 4.3.7. *Clasp moves do not change the number of link components or the number of B-circles in a diagram: If D' is obtained by performing a clasp move on D, then $B_{D'} = B_D$ and $n(D') = n(D)$.*

Proof. Obviously, clasp moves do not change the number of link components in a diagram, and in Figure 4.9 we saw that a clasp moves preserve the number of A-circles. Now in Figure 4.10, we see that clasp moves also preserve the number of B-circles, regardless of if the arcs belonged to the same B-circle or to different B-circles in the original diagram D. □

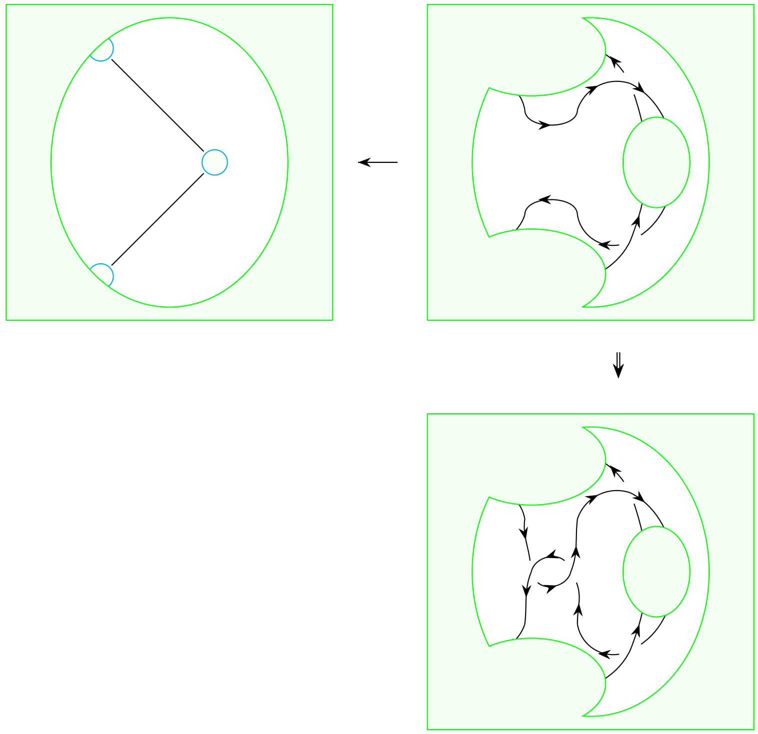

Figure 4.11: A (morally) degree-2 vertex means the adjacent vertices are claspable

4.3.2. Clasp Moves on a Balanced diagram of type 1

Proposition 4.3.8. *Every k-balanced diagram D is claspable.*

Proof. This follows immediately from Definition 4.2.3. □

Recall that by Proposition 4.3.3, every k-balanced diagram has k even and $k \geq 4$.

Proposition 4.3.9. *Let D be a k-balanced diagram. If $k = 4$, then performing a clasp move on a 4-balanced diagram results in a balanced diagram of type 0. If $k \geq 6$, then performing a clasp move on a k-balanced diagram results in a $(k-2)$-balanced diagram.*

Proof. We know that D is claspable. And, we note we can perform a clasp at any (v_1, v_2, v_3) path segment of the cycle in this reduced A-state graph. Since Proposition 4.3.6 tells us that a leaf in the reduced A-state graph of D will correspond to a leaf in the reduced A-state graph of D', it suffices to prove our claim for the case where the reduced A-state graph contains no leaves, and therefore contains only cycle edges. We already saw the case of $k = 4$ demonstrated in Figure 4.8. The proof of the second statement is demonstrated in Figure 4.12, where the dashed edge represents either one edge, or a path of three new edges (adding two vertices in the process, so that all vertices in the dashed path have degree 2). We perform the clasp move, and obtain a $(k-2)$-balanced diagram.

□

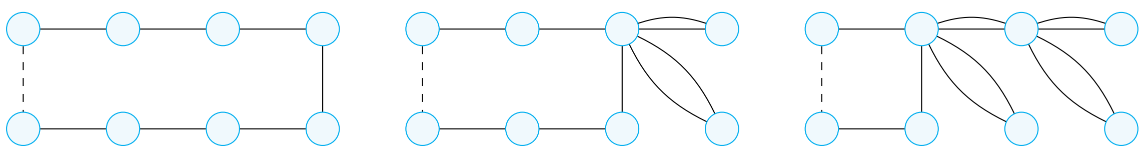

Figure 4.12: Performing successive clasp moves on a k-balanced diagram

Theorem 4.3.10. *Let D be a balanced diagram of type* 1*. Then the number of B-circles in D is equal to the number of link components: $B_D = n$.*

Proof. If D is a balanced type 1 diagram, then it is k-balanced for some even integer k. By induction on k, using the result of Proposition 4.3.9, we have that any k-balanced diagram D can be turned into a balanced diagram D' of type 0 by performing $\frac{k-2}{2}$ clasp moves.

Then

$$\begin{aligned} B_D &= B_{D'} \text{ by Lemma 4.3.7,} \\ &= n(D') \text{ by Theorem 4.3.1, since } D' \text{ is balanced type 0,} \\ &= n(D) \text{ by Lemma 4.3.7.} \end{aligned}$$

□

4.3.3. Clasp Moves on a Balanced diagram of type 2

The general idea will be that if D is a claspable diagram that can be transformed (via clasp moves) into a diagram for which we know that $B = n$, then we have that $B = n$ in D also.

Theorem 4.3.11. *Let D be a balanced diagram of type 2. Then $B_D = n$.*

Proof. Let D be a (k_1, k_2)-balanced diagram. We claim that D is claspable, and that we can choose to perform clasp moves on D such that the resulting diagram D' is also claspable. In doing so we can transform D into a balanced diagram of type 1 or type 0, which we already know has the property that $B = n$.

By definition of balanced type 2, D will satisfy the first two criteria of the claspable definition (Definition 4.3.4). As before, by Proposition 4.3.6 it suffices to prove the rest of our claims on a diagram D whose reduced A-state graph does not contain any leaves.

Let x be the number of edges in the reduced A-state graph that border both holes (interior faces). From the definition of balanced type 2 (Definition 4.2.4), this means we have an even number of edges that are part of cycles, and it follows that x must also be even. We proceed by considering the three cases of $x = 0$, $x = 2$, and $x \geq 4$.

Case: $x = 0$

First, suppose $x = 0$. Then either the reduced A-state graph of D looks like two cycles with one shared vertex v, or it looks like two cycles each with with a vertex (v or w) such that there is a single path connecting them. In either situation, we cut off the second cycle (and the connecting path, if it exists) and focus on the first cycle of length k_1. Let $v = v_1$, and then choose v_2 in either direction along the cycle and we will have a cycle-segment (v_1, v_2, v_3) that satisfies the remaining criteria of being claspable. By performing $\frac{k_1-2}{2}$ clasp moves, we transform the diagram into a balanced diagram of type 0. We get a balanced diagram D' of type 1 when we reattaching the second cycle that we cut off earlier. Meaning, we could have chosen to perform the clasp move at $(v = v_1, v_2, v_3)$ at the beginning, and obtained the very same D' without needing to cut. Since clasp moves do not change the number of the B-circles or the number of link components, by our previous case for balanced type 1 diagrams (Theorem 4.3.10) we see that

$$B_D = B_{D'} = n(D') = n(D).$$

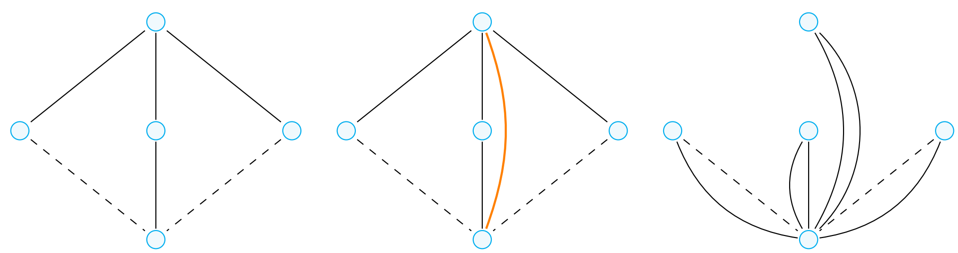

Figure 4.13: $x = 2$: Left is a generalized A-state graph of a balanced type 2 diagram with $x = 2$. The orange in the middle shows the vertices that will be clasped together to produce a diagram whose A-state graph looks like the figure on the right

Case: $x = 2$

Now that $x > 0$, the restriction of the reduced A-state graph having no leaves means that the reduced A-state graph and the A-state graph are identical.

In Figure 4.13, we see on left the general form that every such A-state graph must have, where a dashed edge represents either a single edge or a string of an odd number of edges and even number of vertices, each with degree 2. In the middle, we have connected v_1 and v_3 with an orange line, indicating where our clasp will occur. On the right we show the result of clasping.

If each dashed line represents one edge each, the graph on the right corresponds to a balanced diagram of type 0. If exactly one of the dashed lines represents a path of at least three edges, then the graph on the right corresponds to a balanced diagram of type 1. If both of the dashed lines represent paths of at least three edges, then the graph corresponds to a balanced diagram of type 2 in which $x = 0$. We have already shown that for any of these possibilities, we have $B = n$. So since clasp moves do not change the number of link components or the number of B-circles (Proposition 4.3.7), it follows that $B = n$ in our original diagram D, corresponding to the graph on the left.

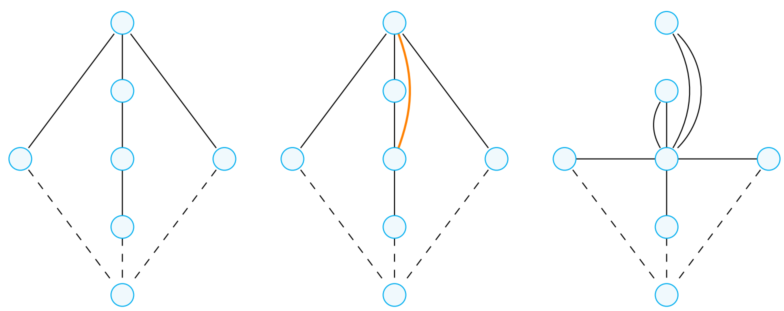

Figure 4.14: $x \geq 4$: The orange in the middle shows the vertices that will be clasped together to produce a diagram whose A-state graph looks like the figure on the right

Case: $x \geq 4$

We see in Figure 4.14 that we can choose to perform a clasp move on the diagram that will transform it into a balanced diagram of type 2, and have x has decreas by 2 in the process. So, we can perform $\frac{x-2}{2}$ total clasp moves to transform it into a balanced diagram of type 2 in which $x = 2$. By our usual clasping argument, this means that $B = n$ for balanced diagrams of type 2 when $x \geq 4$.

And thus $B = n$ for any balanced diagram of type 2. □

4.3.4. Clasp Moves on an Oddly Balanced diagram of type 2

We proceed in much the same manner for oddly balanced diagrams. However, we run into an obstacle. In Figure 4.15 we see a diagram of knot 7_4 that is an oddly balanced diagram. But this is a diagram with 1 link component and 3 B-circles, so clearly we cannot say that $B = n$ for oddly balanced diagrams. However, we will see that this

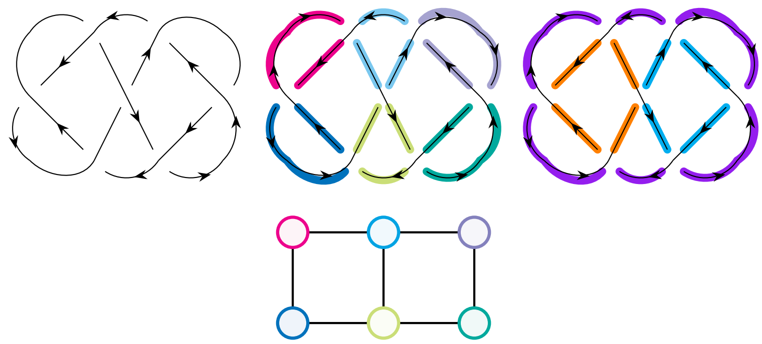

Figure 4.15: An oddly balanced knot diagram (left), its A-circles (top middle), A-state graph (bottom), and its B-circles (right)

is as bad as it gets, and while the number of B-circles might not always be equal to the number of link components, it will only ever be a little bit off-balance.

Theorem 4.3.12. *Let D be an oddly balanced diagram of type 2. Then $B = n$ or $B = n \pm 2$.*

Proof. As in the last section, we show that all oddly balanced diagrams of type 2 are claspable, and that by performing a sequence of clasp moves, we can shrink the holes. An oddly balanced diagram has an odd number of cycle edges in its reduced A-state graph, and therefore has an odd number of edges x that bound both holes in the reduced A-state graph. We look at two special cases first, and then look at the three general cases of $x = 1$, $x = 3$, and $x \geq 5$. As before, by Proposition 4.3.6 it suffices to look at the A-state graphs of diagrams D whose reduced A-state graphs have no leaves. (That is, it suffices to look at A-state graphs G where every edge is part of a cycle.)

Two Special Cases

In Figure 4.16 we see a graph G that is the A-state graph of a $(4, 4)$-oddly balanced diagram D. While it is claspable, we see that any choice of where to clasp the diagram results in a burdened diagram D' of type 1 with one extra edge preventing it from

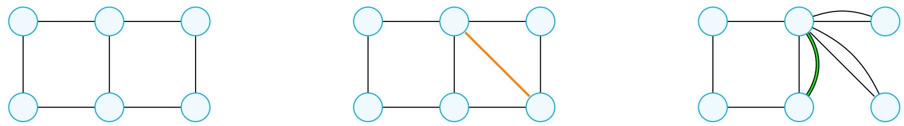

Figure 4.16: Special case $x = 1$: We have only one (up to symmetry) choice of where to clasp, and this gives us a burdened diagram of type 1, with edge preventing it from being balanced overlined in green.

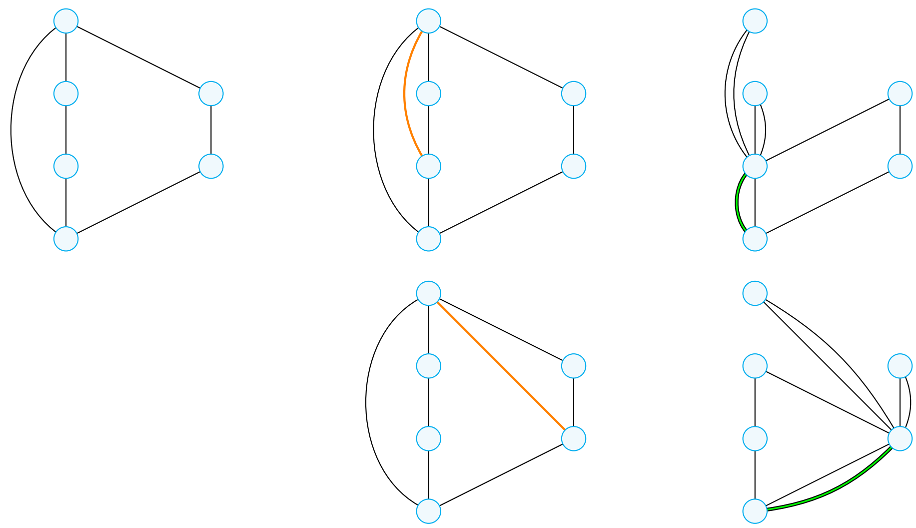

Figure 4.17: Special Case $x = 3$: The middle column shows our choice of clasping, and the right column shows the diagram that results from performing that clasp. The extra edge preventing the diagram from being balanced is overlined in green.

being balanced. By Corollary 4.4.2 we have that $B_{D'} = n(D')$ or $B_{D'} = n(D') \pm 2$. Since clasp moves do not change the number of link components or the number of B-circles, this means that $B_D = B_{D'} = n(D') = n(D)$, or $B_D = B_{D'} = n(D') \pm 2 = n(D) \pm 2$.

In Figure 4.17, we see a graph G that is the A-state graph of a $(4, 6)$-oddly balanced diagram D. While it is claspable, any choice of where to clasp the diagram results in a burdened diagram of type 1 with one edge preventing it from being balanced. In the same manner as the first special case, it follows by Corollary 4.4.2 that the number of B-circles in the resulting diagram D' differs from the number of link components by at most 2. And the result follows.

<u>Case: $x = 1$</u>

If $x = 1$ and (without loss of generality) $k_2 \geq 6$, then we can perform a clasp move to transform the (k_1, k_2)-oddly balanced diagram into a $(k_1, k_2 - 2)$-oddly balanced diagram. As in the case of balanced type 2 diagrams, we see (Figure 4.18) that the resulting diagram is still claspable. We can perform $\frac{k_2-4}{2}$ clasp moves to tranform the diagram into a $(k_1, 4)$-oddly balanced diagram, and then perform $\frac{k_1-4}{2}$ clasp moves to transform it into a $(4, 4)$-oddly balanced diagram, D'. Then D had the same number of link components and B-circles as D', and we just showed that D' has $B = n$ or $B = n \pm 2$, so the result follows.

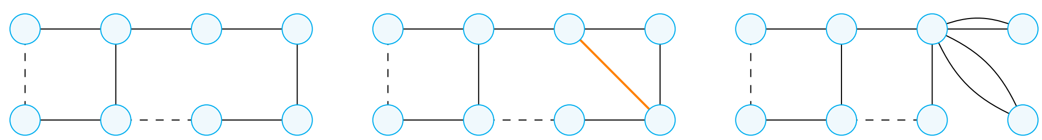

Figure 4.18: $x = 1$: When $k_2 \geq 6$, we can clasp our (k_1, k_2)-oddly balanced diagram and obtain a $(k_1, k_2 - 2)$-oddly balanced diagram

<u>Case: $x = 3$</u>

If $x = 3$ and $k_1, k_2 \geq 6$, then we can perform a clasp move (shown in Figure 4.19) that transforms the diagram into a $(k_1 - 2, k_2 - 2)$-oddly balanced diagram with $x = 1$. Whatever value we have for $k_1 - 2$ and $k_2 - 2$, we have already dealt with it in the previous cases, and the result follows.

<u>Case: $x = 5$</u>

If $x = 5$, observe that at least one of k_1, k_2 is $\geq x + 1$. (If they were both equal to $x + 1$, then we would have a repeated edge that is part of a nontrivial cycle, which violates the definition of an oddly balanced diagram.) Figure 4.20 shows a how we can perform a clasp move to transform the diagram into a $(k_1 - 2, k_2 - 2)$-oddly

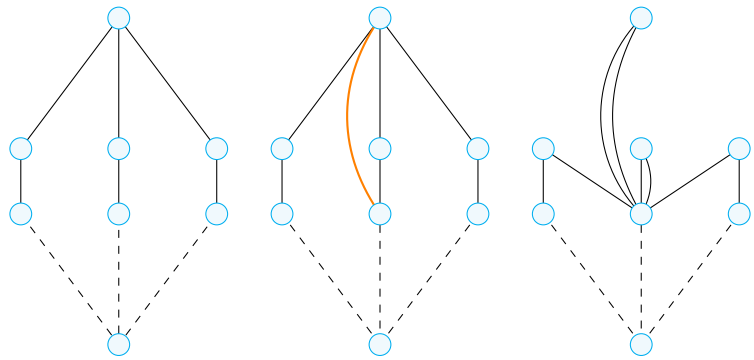

Figure 4.19: $x = 3$: The orange in the middle shows the vertices that will be clasped together to produce a diagram whose A-state graph looks like the figure on the right

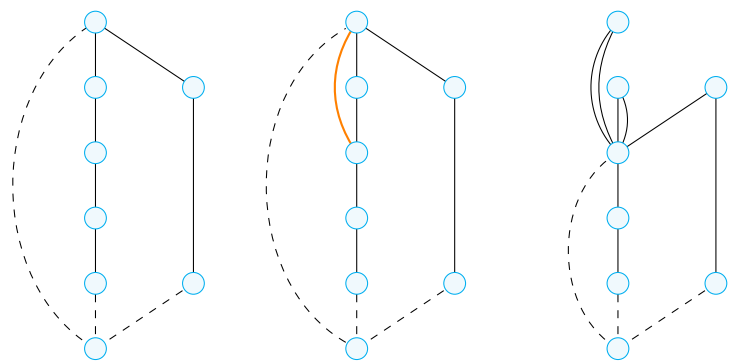

Figure 4.20: $x \geq 5$: The orange in the middle shows the vertices that will be clasped together to produce a diagram whose A-state graph looks like the figure on the right

balanced diagram in which x has decreased by 2. If we instead performed $\frac{x-3}{2}$ clasp moves of this form, then we get an oddly balanced diagram with $x = 3$, and this was the previous case. Again, since clasp moves do not change the number of link components or the number of B-circles, the result follows.

Now we have proven the result for all oddly balanced diagrams of type 2.

□

Section 4.4

Burdened and Oddly Burdened Diagrams

In every burdened (or oddly burdened) diagram of type $0, 1$, or 2, we can smooth away some crossings to obtain a balanced (or oddly balanced) diagram of type $0, 1$, or 2. In Figure 4.21 we see an example in which smoothing just one crossing transforms a burdened diagram of type 1 into a balanced diagram of type 0. In general, we may not always be able to quickly obtain a balanced diagram of a smaller type. What we *can* always do, though, is smooth away some of the crossings of a burdened (or oddly burdened) diagram of type r to obtain a balanced (or oddly balanced, respectively) diagram of the same type r.

Definition 4.4.1. The number of crossings that must be smoothed away in a burdened (or oddly burdened) diagram of type r to produce a balanced (or oddly balanced) diagram of the same type r is called the **burdening number**. It counts the number of crossings that burden the diagram, preventing it from being a balanced (or oddly balanced) diagram of the same type. We denote the burdening number by m.

The burdening number is the least upper bound on the number of crossings that must be smoothed in order to obtain a balanced (or oddly balanced) diagram of any type. It is also the greatest number of crossings than can be smoothed away to result in a balanced (or oddly balanced) diagram.

Corollary 4.4.2 (Corollary to Theorems 4.3.10 and 4.3.11)**.** *Let D be a burdened diagram of type 1 or 2. Let B_D be the number of B-circles in the diagram, let $n(D)$ be the number of link components, and let m be the burdening number. Then*

$$B_D \leq n(D) + 2m.$$

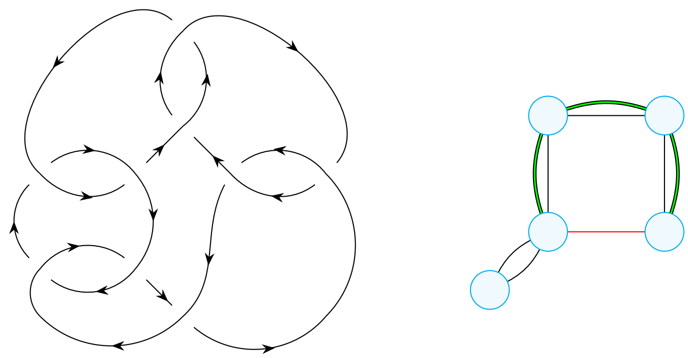

Figure 4.21: Smoothing the crossing that corresponds to the red edge will transform this burdened type 1 diagram into a balanced type 0 diagram. Smoothing crossings corresponding to the three green edges will transform this burdened type 1 diagram into a balanced type 1 diagram. The burdening number is $m = 3$.

Proof. Let D be a burdened diagram of type r. Smooth m crossings to obtain a balanced diagram D' of the same type r. Each smoothing changes the number of B-circles in the diagram by ± 1 and changes the number of link components by ± 1. So, $B_D \leq B_{D'} + m$ and $n(D) \leq n(D') + m$. (This follows from Proposition 2.9 in [2].) Then by Theorems 4.3.10 and 4.3.11, we can say that

$$\begin{aligned} B_D &\leq B_{D'} + m \\ &= n(D') + m \\ &\leq n(D) + 2m. \end{aligned}$$

□

Corollary 4.4.3. *Let D be an oddly burdened diagram of type 2. Let B_D be the number of B-circles, let $n(D)$ be the number of link components, and let m be the burdening number. Then*

$$B_D \leq n(D) + 2(m + 1).$$

Proof. By the same argument as the proof of Corollary 4.4.2, but using Theorem

4.3.12 for oddly balanced diagrams, we can say that

$$\begin{aligned} B_D &\leq B_{D'} + m \\ &= \begin{cases} n(D') + 2 + m \\ n(D') + m \\ n(D') - 2 + m \end{cases} \\ &\leq n(D) + 2(m+1). \end{aligned}$$

□

Corollary 4.4.4. *For any burdened type* $0, 1,$ *or* 2 *diagram* D *of a link* L *with* n *link components and burdening number* m*,*

$$\max\deg V_L \leq c(D) + \frac{n-1}{2} + m.$$

For any oddly burdened type 2 *diagram* D*,*

$$\max\deg V_L \leq c(D) + \frac{n-1}{2} + m + 1.$$

Proof. For burdened diagrams, the follows directly from 2.2.5 and Corollary 4.4.2:

$$\max\deg V_L \leq c(D) + \frac{B_D - 1}{2} \leq c(D) + \frac{n + 2m - 1}{2} = c(D) + \frac{n-1}{2} + m.$$

For oddly burdened diagrams, the result follows directly from 2.2.5 and Corollary 4.4.3:

$$\max\deg V_L \le c(D) + \frac{B_D - 1}{2} \le c(D) + \frac{n + 2(m+1) - 1}{2} = c(D) + \frac{n-1}{2} + m + 1.$$

□

We started off with the standard bound on the maximum degree of the Jones polynomial of a link with positive diagram D: $\max\deg V \le c(D) + \frac{B_D - 1}{2}$ (2.2.5). For links with balanced diagrams, we now have replaced this diagram-dependent quantity B_D with the diagram-independent n. But we still have the diagram-dependent quantities $c(D)$ and m to contend with. In [2], we found that in every burdened diagram of type 0, we can express the burdening number as $m = 4\min\deg V - c(D)$. This allowed us to replace $c(D)$ and m with the diagram-independent quanitity $4\min\deg V$ and find a bound on the maximum degree of the Jones polynomial for fibered positive links.

In the following subsections, we find similar expressions for the burdening number of burdened diagrams of type 1, burdened diagrams of type 2, and oddly burdened diagrams of type 2. We will see that these expressions allow us to replace $c(D)$ and m in the bound given by Corollary 4.4.4 with diagram-independent quantities.

4.4.1. Burdened Diagrams of Type 1

Proposition 4.4.5. *Let D be a k-burdened diagram of a link L. Then the burdening number m (the number of crossings that must be smoothed to transform D into a k-balanced diagram) can be expressed as*

$$m = 4\min\deg V_L - c(D) + k - 2.$$

Proof. We first consider a k-balanced diagram D. Let G be the A-state graph of a k-balanced diagram D and G' be the reduced A-state graph of D. Since D is k-

balanced, there is a hole in G' that is bounded by k edges. Since G' is a graph with one cycle, the number of edges in G' is $A_D - 1 + 1 = A_D$. By definition of balanced, every edge in G' that is part of the cycle corresponds to one edge in G, and any other edge in G' corresponds to two edges in G. Thus the number of edges in G is $k + 2(A_D - k) = 2A_D - k$. As the number of crossings in D is equal to the number of edges in G, this means that

$$c(D) = 2A_D - k. \tag{4.4.6}$$

Now, let D be a k-burdened diagram with m crossings that must be smoothed to obtain k-balanced D'. As such, we know that $c(D) = c(D') + m$. Since smoothing those crossings does not change the number of A-circles, we also know that $A_D = A_{D'}$. It follows from 4.4.6 that

$$c(D) = 2A_D - k + m. \tag{4.4.7}$$

Since D is a positive diagram, we have from 2.2.4 that $4 \min\deg V_L = 2(c(D) - A_D + 1)$. It follows from 4.4.7 that

$$\begin{aligned} m &= c(D) - 2A_D + k \\ &= 4 \min\deg V_L - c(D) + k - 2. \end{aligned}$$

□

Putting this information into the bound in Corollary 4.4.4 will eradicate $c(D)$ and m, but will introduce another diagram-dependent quantity: k. However, in section 4.5.1 we prove that

Lemma (4.5.15). *For a link L with a k-burdened diagram D and Conway polynomial*

∇_L,

$$\text{lead coeff } \nabla_L = \frac{k}{2}.$$

So, we can replace the diagram-dependent quantity k with the diagram-dependent quanitity $2 \,\text{lead coeff } \nabla_L$.

Theorem 4.4.8. *Let D be a burdened diagram of type 1. Then*

$$\max\deg V_D \leq 4 \min\deg V_D + \frac{n-1}{2} + 2 \,\text{lead coeff } \nabla_D - 2,$$

where V_D is the Jones polynomial and ∇_D is the Conway polynomial.

Proof. Let D be a balanced type 1 diagram. Then D is a k-balanced diagram for some k, we can improve our standard bound 2.2.5 as follows:

$$\begin{aligned}
\max\deg V_D &\leq c(D) + \frac{B_D - 1}{2} && \text{(by 2.2.5)} \\
&= c(D) + \frac{n-1}{2} + m && \text{(by Corollary 4.4.4)} \\
&= c(D) + \frac{n-1}{2} + 4\min\deg V_L - c(D) + k - 2 && \text{(by Proposition 4.4.5)} \\
&= 4\min\deg V_L + \frac{n-1}{2} + 2 \,\text{lead coeff } \nabla_L - 2 && \text{(by Lemma 4.5.15).}
\end{aligned}$$

□

4.4.2. Burdened (and Oddly Burdened) Diagrams of Type 2

This section finds a similar bound on the maximum degree of the Jones polynomial for burdened (and oddly burdened) diagrams of type 2. Using our notation introduced earlier, we are considering (k_1, k_2)-burdened (and oddly burdened) diagrams.

Proposition 4.4.9. *Let D be a $(k_1, k_2)-$burdened diagram. Then m, the number of crossings that must be smoothed to transform D into a $(k_1, k_2)-$balanced diagram, can* *be expressed as*

$$m = 4 \min\deg V_L - c(D) + (k_1 + k_2 - x - 4),$$

where x is the number of of edges shared by the two holes in the reduced A-state graph of D.

Proof. As in the burdened type 1 case, we begin by finding the number of crossings in a balanced (or oddly balanced) diagram first.

Let D be a $(k_1, k_2)-$balanced (or oddly balanced) diagram. Then its reduced A-state graph G' contains exactly 2 holes, where the total number of edges in G' is to $A_D + 1$ (the number of vertices of the graph plus one). One hole is bounded by k_1 edges, the other by k_2 edges, and there are x edges shared between them (where $x \geq 0$). So, the number of edges that are part of a cycle is $k_1 + k_2 - x$, and the number of cut edges is $A_D + 1 - (k_1 + k_2 - x)$.

By definition of balanced (and oddly balanced), every edge that is part of a cycle in the reduced A-state graph corresponds to one crossing in the diagram D, and every cut edge in G' corresponds to exactly 2 crossings in D.

Thus

$$\begin{aligned} c(D) = &\quad 2(\#\text{ cut edges}) \quad + (\#\text{ edges involved in a cycle}) \\ &= 2\Big(A_D + 1 - (k_1 + k_2 - x)\Big) + (k_1 + k_2 - x) \\ &= 2(A_D + 1) - (k_1 + k_2 - x). \end{aligned}$$

That is the number of crossings in a (k_1, k_2)-balanced (or oddly balanced) diagram. Now, let D be (k_1, k_2)-*burdened* diagram. We know it can be smoothed into some balanced (or oddly balanced) diagram D'. Thus $A_{D'} = A_D$ and $c(D) = c(D') + m$, where m is the burdening number. Then

$$c(D) = 2(A_D + 1) - (k_1 + k_2 - x) + m. \tag{4.4.10}$$

Since D is a positive diagram, we have from 2.2.4 that $4 \min \deg V_L = 2(c(D) - A_D + 1)$. It follows from 4.4.10 that

$$\begin{aligned} m &= c(D) - 2A_D - 2 + (k_1 + k_2 - x) \\ &= 4 \min \deg V_L - c(D) + (k_1 + k_2 - x) - 4. \end{aligned}$$

□

Corollary 4.4.11. *Let D be a $(k_1, k_2)-$burdened diagram of with n link components. Then*

$$\max \deg V_D \leq 4 \min \deg V_D + \frac{n-1}{2} + k_1 + k_2 - x - 4,$$

where x is the number of boundary edges shared by both of the two holes in the reduced A-state graph of D. If D is a (k_1, k_2)-oddly burdened diagram with n link components, then

$$\max \deg V_D \leq 4 \min \deg V_D + \frac{n-1}{2} + (k_1 + k_2 - x - 4) + 1.$$

Proof. Let D be a (k_1, k_2)-burdened diagram. Then inserting the information from the above Proposition 4.4.9 and Lemma 4.4.2 into 2.2.5, we obtain:

$$\begin{aligned} \max \deg V_D &\leq c(D) + \frac{B_D - 1}{2} \\ &\leq c(D) + \frac{n + 2m - 1}{2} \text{(by Prop. 4.4.2)} \\ &= c(D) + \frac{n-1}{2} + 4 \min \deg V_L - c(D) + k_1 + k_2 - x - 4 \text{(by Prop. 4.4.9)} \\ &= 4 \min \deg V_L + \frac{n-1}{2} + k_1 + k_2 - x - 4. \end{aligned}$$

□

If instead D is a (k_1, k_2)-oddly burdened diagram, then Proposition 4.4.3 tells us that $B_D \leq 2(m+1)$ (instead of $B_D \leq 2m$ as in the burdened case), and the result for oddly burdened follows from the same argument.

Theorem 4.4.12. *Let D be a (k_1, k_2)-burdened diagram with Conway polynomial ∇. Then*

$$\text{lead coeff } \nabla = \frac{k_1 k_2 - x^2}{4},$$

where x is the number of edges that bound both of the holes in the reduced A-state graph.

Theorem 4.4.13. *Let D be a (k_1, k_2)-oddly burdened diagram with Conway polynomial ∇. Then*

$$\text{lead coeff } \nabla = \frac{k_1 k_2 - x^2 + 1}{4},$$

where x is the number of edges that bound both of the holes in the reduced A-state graph.

The proof of these appears in section 4.5.2.

Lemma 4.4.14. *Let D be a (k_1, k_2)-burdened diagram or a (k_1, k_2)-oddly burdened diagram. Then*

$$\max\deg V \leq 4 \min\deg V + \frac{n-1}{2} + \text{lead coeff } \nabla.$$

Proof. <u>Case 1: Burdened diagram</u>

Let D be a (k_1, k_2)-burdened diagram.

By Corollary 4.4.11 and Theorem 4.4.12, it suffices to show that

$$k_1 + k_2 - x - 4 \leq \frac{k_1k_2 - x^2}{4}. \tag{4.4.15}$$

To aid our computations, we make the following substitutions: let $y = k_1 - x$ and $z = k_2 - x$. Observe that then x, y, z are all even. We claim that

$$\begin{aligned} 0 &\leq xy + xz + yz - 4(x + y + z) + 16 \\ &= (x-2)(y-2) + (x-2)(z-2) + (y-2)(z-2) + 4. \end{aligned} \tag{4.4.16}$$

The equality on the second line is clear, and it remains to show the inequality on the first line. At most one of x, y, z can be 0. If $x = 0$, then $y \geq 4$ and $z \geq 4$, so the right side of 4.4.16 is $yz - 4(y + z) + 16 = (y - 4)(z - 4) \geq 0$, as desired. If none of x, y, z are 0, then all are ≥ 2, so the right side second line of 4.4.16 is at least 4, and the inequality is satisfied.

Inequality 4.4.16 is equivalent to

$$x + y + z - 4 \leq \frac{xy + xz + yz}{4}, \tag{4.4.17}$$

which, via our substitution, is exactly Inequality 4.4.15, and thus we have proved the burdened case.

Case 2: Oddly Burdened diagram

Let D be a (k_1, k_2)-oddly burdened diagram. By Corollary 4.4.11, it suffices to show that

$$(k_1 + k_2 - x - 4) + 1 \leq \frac{k_1k_2 - x^2 + 1}{4}. \tag{4.4.18}$$

As before, we make the following substitutions: let $y = k_1 - x$, and let $z = k_2 - x$. Observe that in this case, x, y, z are all odd. We claim that

$$\begin{aligned} 0 &\leq xy + xz + yz - 4(x + y + z) + 13 \\ &= (x-2)(y-2) + (x-2)(z-2) + (y-2)(z-2) + 1. \end{aligned} \tag{4.4.19}$$

The equality on the bottom line is clear, and we must prove the inequality on the top. At most one of x, y, z can be 1, otherwise we would have a "hole" bounded in the reduced A-state graph that is by only two edges, which is impossible (this would mean the graph had duplicate edges). If $x = 1$, then $y \geq 3$ and $z \geq 3$, so the right side of 4.4.19 is $y + z + yz - 4(1 + y + z) + 13 = (y - 3)(z - 3) \geq 0$, as desired. If none of x, y, z are 1, then all are ≥ 3, so the right side of 4.4.19 is at least 4, and the inequality is satisfied.

Inequality 4.4.19 is equivalent to

$$(x + y + z - 4) + 1 \leq \frac{xy + xz + yz + 1}{4}, \tag{4.4.20}$$

which, via our substitution, is exactly Inequality 4.4.18, and thus we have proved the oddly burdened case.

□

Section 4.5

Conway Polynomial

In this section, we gather whatever tools we can about positive and positive links, in preparation to find the leading Conway coefficient of a generic burdened link. We build up resources here that will always allow us to "prune away" any leaves (or other foliage), and instead find that we can compute the leading Conway coefficient just from the parts of the diagram that correspond to cycle edges in the reduced A-state graph.

Proposition 4.5.1 (Cromwell)**.** *[4] Let L be a positive link with positive diagram D. Then*

$$1 - \chi(D) = 1 - \chi(L) = \max\deg \nabla_L = 2\min\deg V_L.$$

Proposition 4.5.2 (Cromwell)**.** *[4] Positive links have positive Conway polynomials, and almost-positive links have positive Conway polynomials.*

Proposition 4.5.3 (Stoimenow, [18])**.** *Let L be an almost-positive link. Then*

$$\max\deg \nabla_L = 2\min\deg V_L = 1 - \chi(L)$$

The following propositions and theorems involve considering two types of almost-positive diagrams: one type in which the negative crossing and a positive crossing both connect the same pair of Seifert circles, and another type in which no positive crossing connects the same pair of Seifert circles as the negative crossing. Discussion of these two separate situations appears in the work of Feller, Lewark, and Lobb ([5], notions of *parallel crossings* and *type I and type II diagrams*); Ito and Stoimenow ([16, 10] notions of *Seifert equivalent crossings* and *type I and type II diagrams*, and *good and bad crossings* and *good successively k-almost positive diagrams*); and Tagami ([20]). We recall that for positive crossings, performing an A-smoothing is the same as smoothing according to Seifert's algorithm, so in a positive the A-state circles are exactly the same as the Seifert circles.

Lemma 4.5.4 (Stoimenow, [10])**.** *Let L be a link represented by an almost-positive diagram D with negative crossing q. If D is of type I (there is no other crossing p which connects the same pair of Seifert circles as q), then $\chi(L) = \chi(D)$. If D is of* *type II (there is another crossing p which connects the same pair of Seifert circles as q), then* $\chi(L) - 2 = \chi(D) < \chi(L)$.

Remark 4.5.5. We note that then the first case means (for a positive diagram D_+ related to an almost-positive diagram D_- by one crossing change) that $2 \min\deg V(L_+) = 1 - \chi(L_+) = 1 - \chi(D_+) = 1 - \chi(D_-) = 1 - \chi(L_-)$, which (by Proposition 4.5.3 if the link is almost-positive, or by Proposition 4.5.1 if the link is positive) is equal to $2 \min\deg V(L_-)$, so $\min\deg V(L_+) = \min\deg V(L_-)$. And in the other case, we have that $\min\deg V(L_+) = \min\deg V(L_-) + 1$.

Proposition 4.5.6. *Let D_+ be a positive diagram with one distinguished crossing q, and let D_- be the result of making q negative. If there is another crossing in D_+ connecting the same two A-circles as q, then*

$$\deg \nabla_- < \deg \nabla_+.$$

Proof. Recall the Conway skein relation: $\nabla_+ - \nabla_- = z\nabla_0$.

If D_+ is a positive diagram, and D_- is the result of changing one crossing to be negative, and D_0 is the result of smoothing that crossing, then the skein relation and 4.5.2 tells us that

$$\deg \nabla_- \leq \deg \nabla_+.$$

Let D_+ be a positive diagram, D_- the result of changing one crossing (q) to be negative, and D_0 the result of smoothing q.

Since D_+ and D_0 are positive, we have that (where n is the number of link componets in D_+, and $* = \min\deg V_+ = \frac{c(D_+) - s(D_+) + 1}{2} = 1 - \chi(D_+)$, and r is the absolute value of its second Jones coefficient):

- $V_+ = (-1)^{n+1}\Big(t^* - (r)t^{*+1} + \dots\Big)$ where the rest of the terms are of higher degree

- If another crossing in D_+ connects the same pair of Seifert circles as q, then smoothing q does not change the number of holes in the reduced A-state graph and so $V_0 = (-1)^n \Big(t^{*-1/2} - (r) t^{*+1/2} + \dots \Big)$ where the rest of the terms are of higher degree

Recall the skein relation: $t^{-1}V_+ - tV_- = (t^{1/2} - t^{-1/2})V_0$,

And consider the following:

$$\begin{aligned}
(t^{1/2} - t^{-1/2})V_0 &= (-1)(t^{-1/2} - t^{1/2})V_0 \\
&= (-1)^{n+1}\Big(t^{*-1} - (r)t^* - t^* + (r)t^{*+1} + \dots \Big) \qquad (4.5.7) \\
&= \underbrace{(-1)^{n+1}\Big(t^{*-1} - (r+1)t^* + (r)t^{*+1} + \dots \Big)}_{:= **}
\end{aligned}$$

and then by the skein relation,

$$\begin{aligned}
V_- &= t^{-1}\Big(t^{-1}V_+ - (t^{1/2} - t^{-1/2})V_0 \Big) \\
&= t^{-1}\Bigg((-1)^{n+1}\Big(t^{*-1} - (r)t^* + \dots \Big) - (**) \Bigg) \\
&= t^{-1}(-1)^{n+1}\Big(t^* - (r)t^{*+1} + \dots \Big)
\end{aligned}$$

And so $\min\deg V_- = * - 1 = (\min\deg V_+) - 1$.

Thus we have that

$$\underbrace{\deg \nabla_- = 2\min\deg V_-}_{\text{by Thm. 4.5.3}} = 2\min\deg V_+ - 2 < \underbrace{2\min\deg V_+ = \deg\nabla_+}_{\text{by Thm. 4.5.1}}.$$

□

Remark 4.5.8. The remaining terms not explictly written out in line 4.5.7 are of degree $* + 1$ or higher, and this trickles down to the final line. So we do not claim that the second coefficient of V_- must be equal to $\pm r$.

The preceeding argument is mentioned in Stoimenow's work in the proofs for Proposition 4.5.3 and Lemma 4.5.4 [16].

As we will see, what this means for us is that if two crossings connect the same pair of A-circles, we can smooth one of them away and not change the degree or leading coefficient of the Conway polynomial. This also appears in the work of Stoimenow and Ito [10].

Corollary 4.5.9. *Let D_+ be an almost-positive diagram with one distinguished crossing q, let D_- be the result of making q negative, and let D_0 be the result of smoothing it. If there is another crossing in D_+ connecting the same two A-circles as q, then*

$$\text{lead term } \nabla_+ = z \text{ lead term } \nabla_0.$$

Proof. The Conway skein relation tells us that $\nabla_+ = \nabla_- + z\nabla_0$. D_+ and D_0 are both positive diagrams, have the same number of A-circles and their crossing number differs by 1, so by Proposition 4.5.1,

$$\deg \nabla_+ = c(D_+) - A_{D_+} + 1 = c(D_0) - A_{D_0} + 1 + 1 = \deg \nabla_0 + 1.$$

That is, ∇_0 definitely contributes to the leading term of ∇_+. What about ∇_-? Since D_+, D_-, and D_0 are all positive or almost-positive diagrams, they all represent positive or almost-positive links, and therefore by Proposition 4.5.2, all of ∇_+, ∇_-, and ∇_0 are positive. It follows that if $\deg \nabla_- < \deg \nabla_+$, then $\text{lead term } \nabla_+ = z \text{ lead term } \nabla_0$. Similarly, if $\deg \nabla_- = \deg \nabla_+$, then $\text{lead term } \nabla + = \text{lead term } \nabla_- + z \text{ lead term } \nabla_0$.

By Proposition 4.5.6, we are done.

□

Lemma 4.5.10. *Let D be a positive diagram. Let D_m be the result of adding m crossings D such that at every intermediate diagram D_i (for $0 \leq i \leq m$), the reduced A-state graph of D_i is exactly the reduced A-state graph of D. Then*

$$\text{lead term}\, \nabla_{D_m} = z^m \,\text{lead term}\, \nabla_D.$$

Proof. We proceed by induction on m. If $m = 0$, then $D_0 = D$ and we are done.

If $m = 1$, then we have added a single crossing q to D to create D_1, and we have not changed the underlying reduced A-state graph structure. Therefore, there exists another crossing p in D_1 that connects the same pair of A-circles as q. By Corollary 4.5.9, and letting $D_+ = D_1$ and $D_0 = D$, we have that

$$\text{lead term}\, \nabla_{D_1} = z \,\text{lead term}\, \nabla_D$$

.

Now assume there is some value of m for which the statement holds, and consider a diagram D_{m+1}. This is the result of adding $m + 1$ crossings to D, so is also the result of adding 1 crossing to D_m while preserving the underlying graph structure. Therefore, by the same argument as in the base case,

$$\text{lead term}\, \nabla_{D_{m+1}} = z \,\text{lead term}\, \nabla_{D_m}.$$

By inductive hypothesis, $\text{lead term}\, \nabla_{D_m} = z^m \,\text{lead term}\, \nabla_D$, so in total

$$\text{lead term}\, \nabla_{D_{m+1}} = z\Big(z^m \,\text{lead term}\, \nabla_D\Big) = z^{m+1} \,\text{lead term}\, \nabla_D.$$

□

Lemma 4.5.11. *Let D_m be:*

- *a burdened diagram of type* 1 *in which* m *crossings can be smoothed to obtain balanced type* 1 *diagram* D*,*
- *a burdened diagram of type* 2 *in which* m *crossings can be smoothed to obtain a balanced type* 2 *diagram* D*, or*
- *an oddly burdened diagram of type* 2 *in which* m *crossings can be smoothed to obtain an oddly balanced type* 2 *diagram* D*.*

Then

$$\text{lead term}\,\nabla_{D_m} = z^m\,\text{lead term}\,\nabla_D.$$

Proof. This follows immediately from Lemma 4.5.10. □

So, to find the leading coefficient of the Conway polynomial of burdened diagram, it suffices to find the leading coefficient of a balanced one.

Lemma 4.5.12. *Let* D *be a positive link diagram with reduced* A*-state graph* G*. Let* G' *be the result of contracting all cut edges in* G*. Then for any positive link diagram* D' *whose reduced* A*-state graph is* G'*,*

$$\text{lead coeff}\,\nabla_{D'} = \text{lead coeff}\,\nabla_D.$$

Proof. Let D be a positive link diagram whose reduced A-state graph G has t cut edges (edges that are not part of cycles). By Lemma 4.5.10, it suffices to consider the case where every edge in G corresponds to exactly one crossing in D.

Then there are t nugatory crossings in D. For each such crossing, we can lift it up and over half of the diagram without changing any other part of the diagram, and the only effect this has is to contract the corresponding edge in G. This new diagram is actually equivalent the old one and thus has the same Conway polynomial, not just the same leading Conway coefficient. As we said before, the result then follows by Lemma 4.5.10. □

Remark 4.5.13. This means that to find the leading Conway coefficient of a positive link D, we can instead:

- Draw its reduced A-state graph G,
- Contract all cut edges to obtain a new graph G',
- Draw a positive link diagram D' whose reduced A-state graph is G' and whose crossings are in one-to-one correspondence with the edges of G', and then
- Find the leading Conway coefficient of D'.

4.5.1. Leading Conway Coefficient for Balanced diagrams of type 1

We begin with the easiest case. The standard alternating diagram of the $T(2, 2p)$ torus link (given positive orientation) is a balanced diagram of type 1. It is known that

Proposition 4.5.14. *The Conway polynomial of the positive torus link $T(2, 2p)$ is $\nabla = pz$.*

Proof. This can be seen using the Conway skein relation, and the observation that smoothing one of the crossings gives the unknot, while changing it gives an almost-positive diagram of the link $T(2, 2p - 2)$. So

$$\nabla_{T(2,2p)} = \nabla_{T(2,2p-2)} + z(1) = \cdots = \nabla_{T(2,2)} + (p - 1)z = pz.$$

□

Lemma 4.5.15. *Let D be a k-burdened diagram. Then the leading coefficient of its Conway polynomial is* $\operatorname{lead\,coeff} \nabla_D = \frac{k}{2}$

Proof. This follows directly from Lemma 4.5.12 and Proposition 4.5.14. □

4.5.2. Leading Conway Coefficient for Balanced diagrams of type 2

As in the last section, it suffices to consider link diagrams whose reduced A-state graphs contain no cut edges. For a graph to be the reduced A-state graph of a balanced type 2 link diagram, x (the number of edges that bound both holes of the graph) must be even.

We will consider the case when $x = 0$ and the case when $x \geq 2$.

Case: $x = 0$

For k_1 and k_2 even and ≥ 4, the connect sum of the standard alternating $T(2, k_1)$ and $T(2, k_2)$ torus link diagrams form a positive link diagram whose reduced A-state graph is such that $x = 0$. The leading Conway coefficient of this composite link is $\frac{k_1}{2}\frac{k_2}{2} = \frac{k_1 k_2}{4}$.

So by Lemma 4.5.12, we just proved:

Proposition 4.5.16. *Let D be a (k_1, k_2)-burdened diagram with reduced A-state graph G. If the two holes of G are not bounded by any of the same edges, then the leading coefficient of its Conway polynomial is*

$$\text{lead coeff } \nabla_D = \frac{k_1 k_2}{4}.$$

Now, we consider the case if there are edges in G that bound both holes.

Case: $x \geq 2$

We observe that the standard alternating diagram of certain pretzel links will have this kind of reduced A-state graph. We also observe that the standard alternating diagram of a positive pretzel link $P(-p, -q, -r)$ with even $p, q, r \geq 2$ is a balanced diagram of type 2. The pretzel with all counter-clockwise twists is the one which actually corresponds to the link in which all crossings are positive. By convention, a counterclockwise twist is indicated with a negative sign, and we choose to follow the standard convention of naming our pretzel links in the following computations.

Lemma 4.5.17. *Let p, q, r even integers ≥ 2. Let $\nabla(-p,-q,-r)$ be the Conway polynomial for the $P(-p,-q,-r)$ positive pretzel link. Then*

$$\nabla(-p,-q,-r) = \left(\frac{pq+pr+qr}{4}\right)z^2.$$

Proof. This follows directly from the Conway skein relation, and the following facts:

- Changing one of the crossings in the p-twist column gives us an almost-positive diagram of the positive link $P(-(p-2),-q,-r)$.
- Smoothing one of the crossings in the p-twist wipes out that whole column, and gives us torus link $T(2, q+r)$, which by Lemma 4.5.14 has Conway polynomial $\nabla = \frac{q+r}{2}z$.
- The $P(0,-q,-r)$ pretzel link is actually the connect sum of torus links $T(2,q)$ and $T(2,r)$. By Lemma 4.5.14 and multiplicativity of the Conway polynomial, this means $\nabla(0,-q,-r) = \nabla(T(2,q)) \cdot \nabla(T(2,r)) = \frac{q}{2}z \cdot \frac{r}{2}z = \frac{qr}{4}z^2$.

Therefore,

$$\begin{aligned}\nabla(-p,-q,-r) &= \nabla(0,-q,-r) + \left(\frac{p}{2}\right)z\nabla(T(2,q+r)) \\ &= \frac{qr}{4}z^2 + \left(\frac{p}{2}\right)\frac{q+r}{2}z^2 \\ &= \frac{pq+pr+qr}{4}z^2.\end{aligned}$$

□

Theorem 4.5.18. *Let D be a (k_1, k_2)-burdened diagram. Then the leading coefficient of its Conway polynomial is*

$$\text{lead coeff } \nabla_D = \frac{k_1 k_2 - x^2}{4}.$$

Proof. We have already proven the case if $x = 0$ in our Proposition 4.5.16.

If $x \geq 2$, then by Lemma 4.5.12 it suffices to find the leading Conway coefficient for any (k_1, k_2)-burdened diagram with that value of x. With our notation for balanced diagrams of type 2, the pretzel link $P(-p, -q, -r)$ for p, q, r even is a $\Big((p+q), (r+q)\Big)$-balanced link, where $k_1 = p + q$, $k_2 = q + r$, and $x = q$.

So, let us put that information into the expression above:

$$\begin{aligned} \frac{k_1 k_2 - x^2}{4} &= \frac{(p+q)(r+q) - (q)^2}{4} \\ &= \frac{pq + pr + qr}{4} \\ &= \text{lead coeff } \nabla(-p, -q, -r) \\ &= \text{lead coeff } \nabla_D, \end{aligned}$$

as desired. □

4.5.3. Oddly Burdened type 2

As in the previous case, we can look to pretzel links for a place to start.

Theorem 4.5.19. *Let D be a (k_1, k_2)-oddly burdened diagram. Then the leading coefficient of its Conway polynomial is*

$$\text{lead coeff } \nabla_D = \frac{k_1 k_1 - x^2 + 1}{4}.$$

Proof. Let D be a (k_1, k_2)-oddly burdened diagram with reduced A-state graph G in which the two holes are bounded by x shared edges, with odd $x \geq 1$.

By our Lemma 4.5.12, we know that the leading Conway coefficient of D will be the same as the leading Conway coefficient of the pretzel link $P(-p,-q,-r)$ where $p = k_1 - x$, $q = x$, and $r = k_2 - x$. (We also note that P is invariant under any permutation of the three arguments, so it really does not matter which value we choose to be x.)

So, it suffices to find that leading coefficient, and show it is equal to the one given above.

Claim 4.5.20. Let p, q, r be odd integers ≥ 1. Let $\nabla(-p,-q,-r)$ be the Conway polynomial for the $P(-p,-q,-r)$ pretzel link, given a positive orientation. Then

$$\nabla(-p,-q,-r) = 1 + \Big(\frac{pq + pr + qr + 1}{4}\Big) z^2.$$

Proof. This follows directly from the Conway skein relation, and the following facts:

- Changing one of the crossings in the p-twist column gives us an almost-positive diagram of the $P(-(p-2),-q,-r)$ pretzel link.
- Smoothing one of the crossings in the p-twist wipes out that whole column, and gives us the $T(2, q+r)$ torus link, which by Lemma 4.5.14 has Conway polynomial $\frac{q+r}{2}$.
- $P(-1,-1,-1)$ is the trefoil, with $\nabla = 1 + z^2$.

Thus

$$\begin{aligned}
\nabla(-p,-q,-r) &= \nabla(-1,-q,-r) + \left(\frac{p-1}{2}\right) z\nabla(T(2,q+r)) \\
&= \nabla(-1,-1,-r) + \left(\frac{q-1}{2}\right) z\nabla(T(2,1+r)) + \left(\frac{p-1}{2}\right)\left(\frac{q+r}{2}\right) z^2 \\
&= \nabla(-1,-1,-1) + \left(\frac{r-1}{2}\right) z\nabla(T(2,2)) + \left(\frac{q-1}{2}\right)\left(\frac{r+1}{2}\right) z^2 \\
&+ \left(\frac{p-1}{2}\right)\left(\frac{q+r}{2}\right) z^2 \\
&= 1 + z^2 + \left(\frac{r-1}{2}\right) z^2 + \left(\frac{q-1}{2}\right)\left(\frac{r+1}{2}\right) z^2 + \left(\frac{p-1}{2}\right)\left(\frac{q+r}{2}\right) z^2 \\
&= 1 + \frac{pq+pr+qr+1}{4} z^2.
\end{aligned}$$

□

Now, let us substitute: $k_1 = p+q$, $k_2 = r+q$, and $x = q$. Then:

$$\begin{aligned}
\frac{k_1k_2 - x^2 + 1}{4} &= \frac{(p+q)(r+q) - (q)^2 + 1}{4} \\
&= \frac{pr + pq + qr + q^2 - q^2 + 1}{4} \\
&= \frac{pq + pr + qr + 1}{4},
\end{aligned}$$

as desired.

□

Section 4.6

Application

In [2], we showed that the positivity obstruction developed for the case of fibered positive knots could be used to show that seven particular knots are not positive, and are instead almost-positive. These were the last remaining knots of crossing number ≤ 12 for which positivity was unknown. Independently and around the same time, Stoimenow compiled a list [19] of all non-alternating positive knots up to 15, also settling the question of those remaining seven knots through a different method.

In Section 4.6.1 we give another example of an almost-positive knot diagram whose non-positivity is proved by failing the test of Theorem 3.1.15. From this 16-crossing diagram, we can construct an infinite family of examples of almost-positive knots, whose non-positivity is shown by failing the test of Theorem 3.1.15 in the exact same way.

In Sections 4.6.2 and 4.6.3 we similarly construct infinite families of knots whose non-positivity is shown by failing the tests of Theorem 4.2.17.

We would like to mention that KnotFolio was an extremely helpful tool for constructing such examples [13].

4.6.1. An Infinite Family of Almost-Positive Knots with Second Jones Coefficient equal to 0

In Figure 4.22, we see a 16-crossing almost-positive link diagram. We can compute the Jones polynomial and the HOMFLY polynomial (provided for the interested reader) with Regina [1], and find that they are:

- Jones polynomial: $V = t^4 - 3t^6 + 12t^7 - 24t^8 + 38t^9 - 49t^{10} + 56t^{11} - 56t^{12} + 48t^{13} - 37t^{14} + 23t^{15} - 12t^{16} + 5t^{17} - t^{18}$
- HOMFLY polynomial $P = \alpha^{-8}z^8 + 8\alpha^{-8}z^6 + 17\alpha^{-8}z^4 + 13\alpha^{-8}z^2 + 3\alpha^{-8} + 3\alpha^{-10}z^6 + 5\alpha^{-10}z^4 - 2\alpha^{-10} + 4\alpha^{-12}z^6 + 10\alpha^{-12}z^4 + 11\alpha^{-12}z^2 + 6\alpha^{-12} - 8\alpha^{-14}z^4 - 18\alpha^{-14}z^2 - 11\alpha^{-14} + 5\alpha^{-16}z^2 + 6\alpha^{-16} - \alpha^{-18}$

The second Jones coefficient is 0. If this diagram represented a positive link, then Theorem 3.1.15 tells us that we would have $\max\deg V \leq 4\min\deg V$. However,

$$\max\deg V = 18 \not\leq 16 = 4\min\deg V,$$

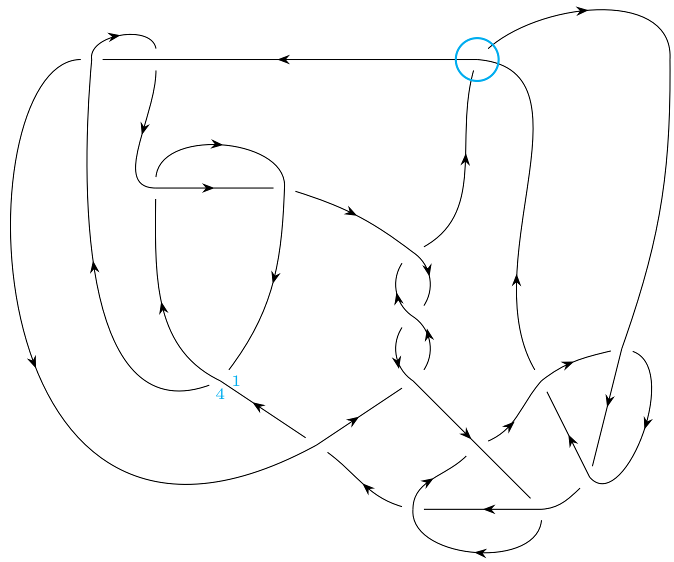

Figure 4.22: A 16-crossing almost-positive diagram (negative crossing circled) of an almost-positive knot with second Jones coefficient equal to 0. DT code: $[4, 8, 22, 2, 26, 24, -30, -12, -28, -16, 6, 32, 10, -20, -18, -14]$

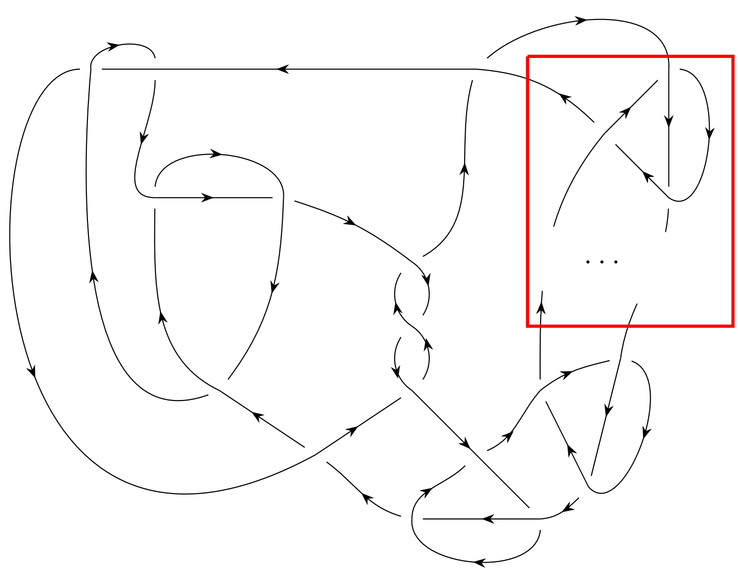

Figure 4.23: An infinite family of almost-positive diagrams of almost-positive knots with second Jones coefficient equal to 0

so the knot (identified by the KnotFinder tool as $16_n125409$ [3]) cannot be positive.

We can use this knot to generate an infinite family of knots that can be shown not to be positive by using Theorem 3.1.15 in the same way. We create a new diagram D_w by adding w copies of a three-crossing loop to the upper left corner of our original diagram D_0, as shown in Figure 4.23, so that the crossing number of our new diagram is $c(D_w) = c(D_0) + 3w$.

Claim 4.6.1. Let $V(D_w)$ be the Jones polynomial of D_w, and let $\nabla(D_w)$ be its Conway polynomial. Then

(a) $\min\deg V(D_w) = \min\deg V(D_0) + w$

(b) $\max\deg V(D_w) = \max\deg V(D_0) + 4w$

(c) The second Jones coefficient of $V(D_w)$ = the second Jones coefficient of $V(D_0)$

If our claims are true, then the second Jones coefficient of D_w is 0, and yet

$$\begin{aligned}
\max\deg V(D_w) &= \max\deg V(D_0) + 4w \\
&= 18 + 4w \\
&\nleq 16 + 4w \\
&= 4\min\deg V(D_0) + 4w \\
&= 4\min\deg V(D_w).
\end{aligned}$$

so by Theorem 3.1.15, D_w cannot be a diagram of a positive knot.

The proof of these is identical to the proofs in the next section.

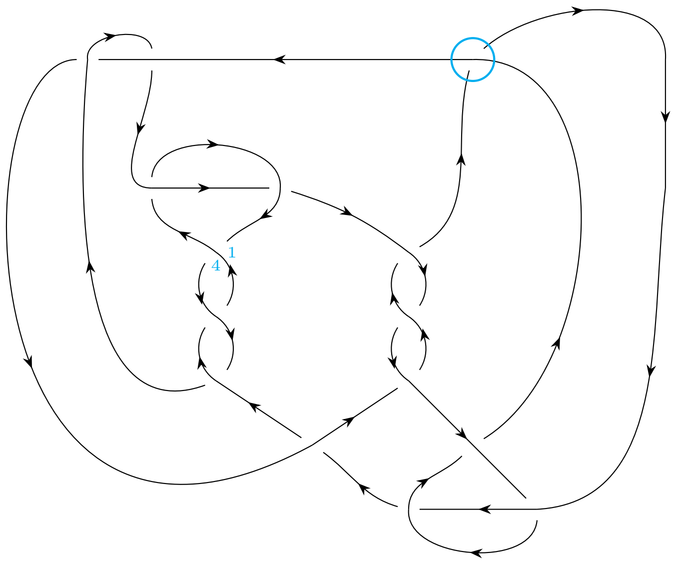

Figure 4.24: A 15-crossing almost-positive knot diagram (negative crossing circled) of an almost-positive knot with second Jones coefficient equal to −1. DT code: [4, 10, 30, 20, 2, 24, 22, −26, −14, 8, 28, 12, −18, −16, 6]

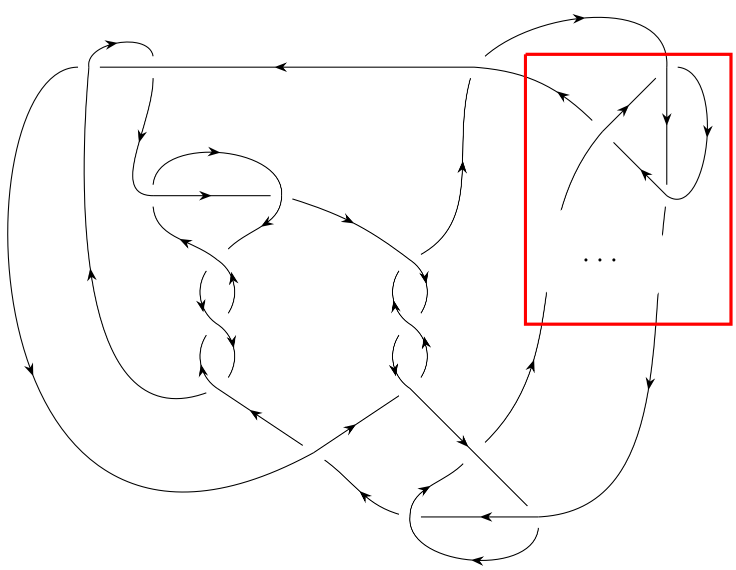

Figure 4.25: An infinite family of almost-positive diagrams of almost-positive knots with second Jones coefficient equal to −1.

4.6.2. An Infinite Family of Almost-Positive Knots with Second Jones Coefficient equal to -1

In Figure 4.24 we see a 15-crossing almost-positive diagram (the negative crossing is circled).

Using the DT code for the pictured diagram

$$[4, 10, 30, 20, 2, 24, 22, -26, -14, 8, 28, 12, -18, -16, 6]$$

and inputting into Regina [1], we obtain the Jones polynomial and the HOMFLY polynomial, from which we can obtain the Conway polynomial under the substitution $P(a = 1, z) = \nabla(z)$.

We have:

- Jones polynomial: $V = t^3 - t^4 + 2t^5 - t^6 - t^7 + 5t^8 - 8t^9 + 11t^{10} - 13t^{11} + 12t^{12} - 10t^{13} + 6t^{14} - 3t^{15} + t^{16}$
- HOMFLY polynomial: $P = a^{-6}z^6 + 5a^{-6}z^4 + 6a^{-6}z^2 + 2a^{-6} + a^{-8}z^6 + 5a^{-8}z^4 + 3a^{-8}z^2 + 2a^{-10}z^4 + a^{-10}z^2 + 2a^{-12}z^4 + a^{-12}z^2 - 3a^{-14}z^2 - 2a^{-14} + a^{-16}$
- Conway polynomial: $\nabla = 1 + 8z^2 + 14z^4 + 2z^6$

Then

$$\max\deg V = 16 \not\leq 14 = 4\min\deg V + 2\,\text{lead coeff}\,\nabla - 2,$$

so by Theorem 4.2.17, this knot (identified by the KnotFinder tool as 15_n11331 [3]) is not positive, and is thus an almost-positive knot, since we do have an almost-positive diagram.

We can use this knot to generate an infinite family of knots that can be shown not to be positive by using Theorem 4.2.17 in the same way. We create a new diagram D_w by adding w copies of a three-crossing loop to the upper left corner of our original diagram D_0, as shown in Figure 4.25, so that the crossing number of our new diagram is $c(D_w) = c(D_0) + 3w$.

Claim 4.6.2. Let $V(D_w)$ be the Jones polynomial of D_w, and let $\nabla(D_w)$ be its Conway polynomial. Then

(a) $\min\deg V(D_w) = \min\deg V(D_0) + w$

(b) $\max\deg V(D_w) = \max\deg V(D_0) + 4w$

(c) The second Jones coefficient of $V(D_w)$ = the second Jones coefficient of $V(D_0)$

(d) lead coeff $\nabla(D_w)$ = lead coeff $\nabla(D_0)$.

If our claims are true, then the second Jones coefficient of D_w is -1, and yet

$$\begin{aligned}\max\deg V(D_w) &= \max\deg V(D_0) + 4w \\ &= 16 + 4w \\ &\not\leq 14 + 4w \\ &= 4\min\deg V(D_0) + 2\,\text{lead coeff}\,\nabla(D_0) - 2 + 4w \\ &= 4\min\deg V(D_w) + 2\,\text{lead coeff}\,\nabla(D_w) - 2.\end{aligned}$$

so by Theorem 4.2.17, D_w cannot be a diagram of a positive knot.

Proof of Claim 4.6.2, part (1).

Proof. Since there is no other crossing in D_w connecting the same pair of Seifert circles as the negative crossing, we have by Stoimenow's Lemma 4.5.4 that $2 \min \deg V(D_w) = 1 - \chi(D_w)$, and thus

$$\begin{aligned} 2 \min \deg V(D_w) &= c(D_w) - s(D_w) + 1 \\ &= (c(D_0) + 3w) - (s(D_0) + w) + 1 \\ &= c(D_0) - s(D_0) + 1 + 2w \\ &= 2 \min \deg V(D_0) + 2w, \end{aligned}$$

So indeed $\min \deg V(D_w) = \min \deg V(D_0) + w$.

□

Proof of Claim 4.6.2, part 2.

Proof. Observe that each D_w, including D_0, is B-adequate, so a result of 2.2.3 says that

$$\begin{aligned} \max \deg V(D_w) &= c(D_w) + \frac{B_{D_w} - 1}{2} - \frac{3}{2} \\ &= (c(D_0) + 3w) + \frac{(B_{D_0} + 2w) - 1}{2} - \frac{3}{2} \\ &= c(D_0) + \frac{B_{D_0} - 1}{2} - \frac{3}{2} + 4w \\ &= \max \deg V(D_0) + 4w. \end{aligned}$$

□

Proof of Claim 4.6.2, parts (3) and (4). Parts (3) and (4) are about the second Jones coefficient and the leading Conway coefficient, and we can get information about both from the HOMFLY(PT) polynomial. Since we need both, we first look at the skein relation of the HOMFLY(PT) polynomial, and then we will specialize to each case.

Proof. Let D_{0S} be the diagram obtained by smoothing the negative crossing in D_0, and let D_{0+} be the diagram obtained by making that crossing positive. Then by the HOMFLY(PT) skein relation, we have

$$\alpha P(D_{0+}) - \alpha^{-1} P(D_0) = zP(D_{0S}). \tag{4.6.3}$$

In Figure 4.26, we look at two "layers" of skein relations. The first layer is D_w with distinguished positive crossing p indicated by the blue arrow. Making p negative will give us a diagram equivalent to D_{w-1}, and smoothing p gives us a diagram equivalent to D_{wS}. For the next layer, we choose one of the two crossings in the top part of D_{wS} shown in Figure 4.26 to be the distinguished crossing, and we see that making that crossing negative gives us a diagram equivalent to $D_{0S}\#3_1\#\dots\#3_1$ (the connected sum of D_{0S} and $w-1$ copies of the trefoil), and smoothing that crossing gives us D_{w-1} back again.

So, the first layer of the skein relation gives us

$$\alpha P(D_w) - \alpha^{-1} P(D_{w-1}) = zP(D_{wS}), \tag{4.6.4}$$

and the second layer gives us

$$\alpha P(D_{wS}) - \alpha^{-1} P(D_{0S}\#3_1\#\dots\#3_1) = zP(D_{w-1}). \tag{4.6.5}$$

Rearranging, multiplying by z, and using the fact that the HOMFLY(PT) polynomial

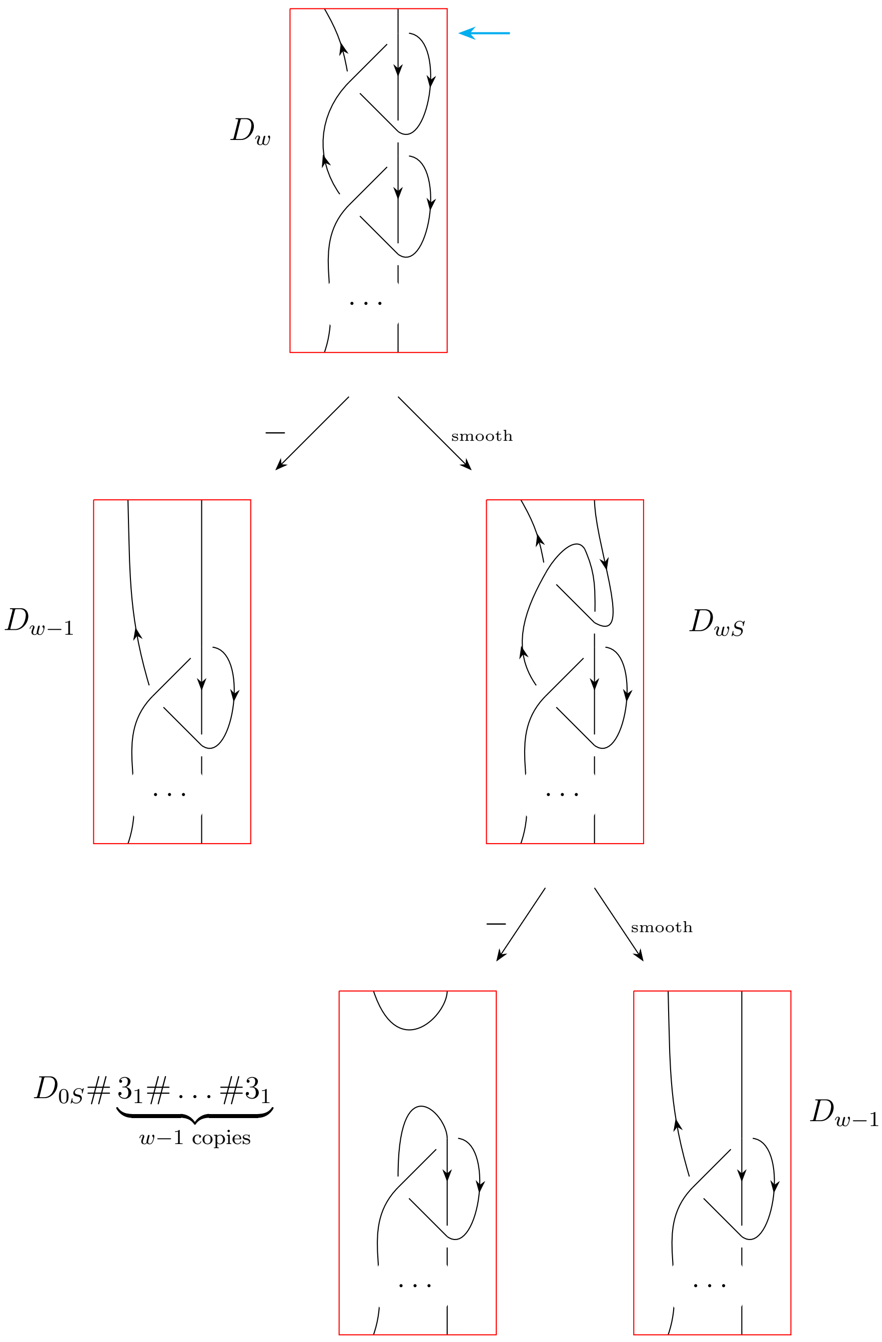
D_w
−
smooth
D_{w-1}
D_{wS}
−
smooth
$D_{0S}\#\underbrace{3_1\#\ldots\#3_1}_{w-1 \text{ copies}}$
D_{w-1}

Figure 4.26:

is multiplicative over connected sums, we get

$$\alpha z P(D_{wS}) = \alpha^{-1} z P(D_{0S}) P(3_1)^{w-1} + z^2 P(D_{w-1}). \tag{4.6.6}$$

We can then express $\alpha^2 P(D_w)$ as:

$$\begin{aligned}
\alpha^2 P(D_w) &= \alpha\Big(\alpha^{-1} P(D_{w-1}) + zP(wS)\Big) && \text{(by 4.6.4)}\\
&= P(D_{w-1}) + \alpha^{-1} z P(D_{0S}) P(3_1)^{w-1} + z^2 P(D_{w-1}) && \text{(by 4.6.6)}\\
&= (1+z^2)P(D_{w-1}) + \alpha^{-1}\Big(\alpha P(D_{0+}) - \alpha^{-1} P(D_0)\Big) P(3_1)^{w-1} && \text{(by 4.6.3)}\\
&= (1+z^2)P(D_{w-1}) + \Big(P(D_{0+}) - \alpha^{-2} P(D_0)\Big) P(3_1)^{w-1} && (4.6.7)
\end{aligned}$$

Now, we specialize to each part.

When we specialize to the Jones polynomial via the substitution $P(\alpha = t^{-1}, z = t^{1/2} - t^{-1/2})$, Equation 4.6.7 becomes

$$t^{-2} V(D_w) = (t^{-1} - 1 + t) V(D_{w-1}) + \Big(V(D_{0+}) - t^2 V(D_0)\Big) V(3_1)^{w-1}. \tag{4.6.8}$$

In particular, for the base case of $w = 1$ we have

$$t^{-2} V(D_1) = (t^{-1} - 1 + t) V(D_0) + \Big(V(D_{0+}) - t^2 V(D_0)\Big) = (t^{-1} - 1 + t - t^2) V(D_0) + V(D_{0+}). \tag{4.6.9}$$

Since there are no (other) crossings in the almost-positive diagram D_0 that connect the same pair of Seifert circles as the negative crossing, it follows from Lemma 4.5.4 that $\min\deg V_{D_0} = \min\deg V_{D_{0+}}$. Therefore, the first term in $V(D_1)$ is contributed entirely by $V(D_0)$, and the second coefficient of $V(D_1)$ is exactly the first coefficient of $V(D_0)$ subtracted from the sum of the second coefficient of $V(D_0)$ and the first coefficient of $V(D_{0+})$. But we know that the first coefficients of both $V(D_0)$ and $V(D_{0+})$ are 1, since D_0 is almost-positive and D_{0+} is positive. Hence the second coefficient of $V(D_1)$ is exactly the second coefficient of $V(D_0)$.

Now assume for induction that the second coefficient of $V(D_w)$ is equal to the second coefficient of $V(D_0)$. We recall that $V(3_1) = t + t^3 - t^4$, and by part (1) we know $\min\deg V(D_w) = \min\deg V(D_0) + w = 3 + w$. Then by 4.6.8, we can write

$$t^{-2}V(D_{w+1}) = \underbrace{(t^{-1} - 1 + t)V(D_w)}_{\min\deg = -1 + \min\deg V(D_0) + w} + \overbrace{\Big(V(D_{0+}) - t^2V(D_0)\Big)V(3_1)^w}^{\min\deg = \min\deg V(D_0) + w}. \tag{4.6.10}$$

Therefore, the first coefficient of $V(D_{w+1})$ is exactly the first coefficient of $V(D_w)$, and the second coefficient of $V(D_{w+1})$ is exactly the first coefficient of $V(D_w)$ subtracted from the sum of the second coefficient of $V(D_w)$ and the first coefficient of $V(D_{0+})$. But again, we know that the first coefficients of both $V(D_w)$ and $V(D_{0+})$ are 1, since D_w is almost-positive and D_{0+} is positive. Hence the second coefficient of $V(D_{w+1})$ is exactly the second coefficient of $V(D_w)$, which is equal to the second coefficient of $V(D_0)$ by our induction hypothesis.

Thus we have that each D_w has the same second Jones coefficient as D_0, which was our third claim.

To prove our last claim, we specialize 4.6.7 to the Conway polynomial, via the substitution $P(\alpha = 1, z)$:

$$\begin{aligned} \nabla(D_w) &= (1 + z^2)\nabla(D_{w-1}) + \Big(\nabla(D_{0+}) - \nabla(D_0)\Big)\nabla(3_1)^{w-1} \\ &= (1 + z^2)\nabla(D_{w-1}) + \nabla(D_{0S})\nabla(3_1)^{w-1}. \end{aligned} \tag{4.6.11}$$

We observe that D_{0S} is a positive diagram with 14 crossings and 10 A-circles, so then $\deg \nabla(D_{0S}) = 14 - 10 + 1 = 5$. And so for the base case of $w = 1$ we have

$$\nabla(1) = \underbrace{(1+z^2)\nabla(D_0)}_{\text{deg}=2+\deg \nabla(D_0)} + \underbrace{z\nabla(D_{0S})}_{\text{deg}=1+5}.$$

So for $w = 1$, since our $\deg \nabla(D_0) = 6$, we then have that $\deg \nabla(D_w) = 2w + \deg \nabla(D_0)$ and $\operatorname{lead\,coeff} \nabla(D_w) = \operatorname{lead\,coeff} \nabla(D_0)$.

We claim that this is true for all $w \geq 1$, and prove it by induction. Suppose our claim is true for w. Then, since $\nabla(3_1) = 1 + z^2$,

$$\nabla(D_{w+1}) = \underbrace{(1+z^2)\nabla(D_w)}_{\text{deg}=2+2w+\deg \nabla(D_0)} + \overbrace{z\nabla(D_{0S})\big(\nabla(3_1)\big)^w}^{\text{deg}=1+5+2w}.$$

Thus $\deg \nabla(D_{w+1}) = 2(w+1) + \deg \nabla(D_0)$, and we must have that $\operatorname{lead\,coeff} \nabla(D_{w+1}) = \operatorname{lead\,coeff} \nabla(D_w)$. And this is equal to $\operatorname{lead\,coeff} \nabla(D_0)$ by our induction hypothesis, so indeed $\operatorname{lead\,coeff} \nabla(D_{w+1}) = \operatorname{lead\,coeff} \nabla(D_0)$.

And we have now proved all four of our claims.

□

4.6.3. An Infinite Family of Almost-Positive Knots with Second Jones Coefficient equal to -2

Figure 4.27 is a 15-crossing almost-positive knot diagram with DT code

$$[4, 8, 22, 2, 20, 26, 24, -28, -14, 10, 6, 30, 12, -18, -16].$$

We use the obstruction provided by Theorem 4.2.17 to prove that this knot cannot be positive.

Using Regina [1], we find its:

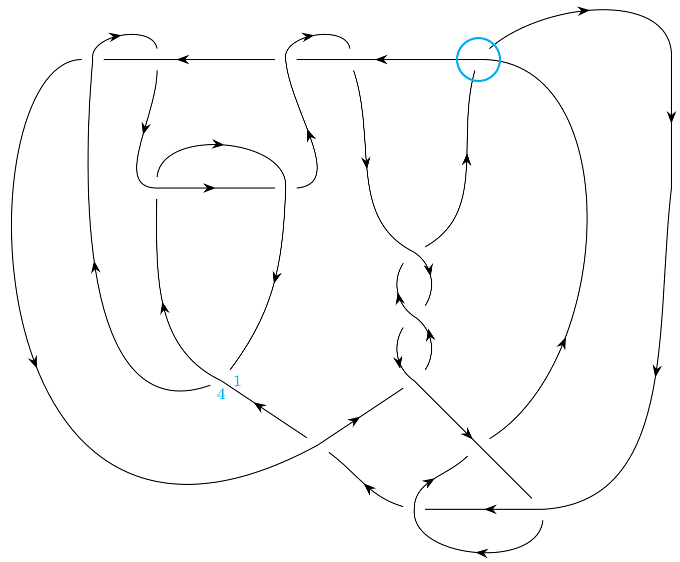

Figure 4.27: A 15-crossing almost-positive knot diagram (negative crossing circled) of an almost-positive knot with second Jones coefficient equal to -2. DT code: $[4, 8, 22, 2, 20, 26, 24, -28, -14, 10, 6, 30, 12, -18, -16]$

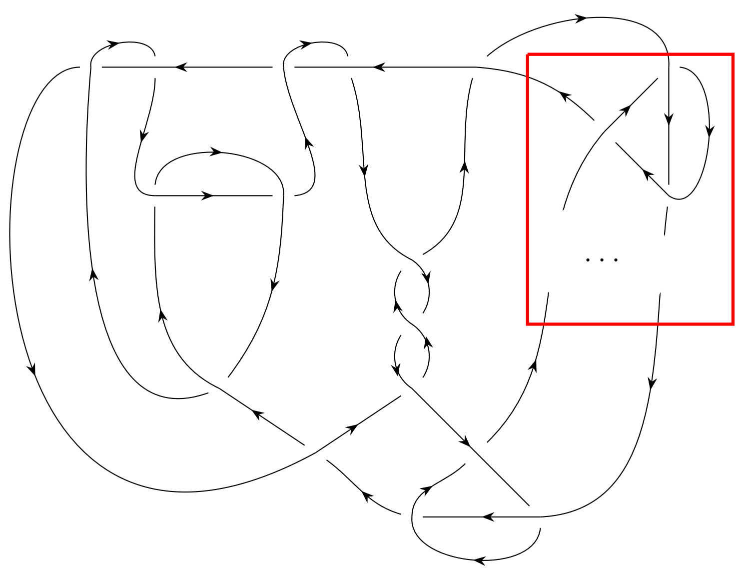

Figure 4.28: An infinite family of almost-positive diagrams of almost-positive knots with second Jones coefficient equal to -2

- Jones polynomial: $V = t^3 - 2t^4 + 5t^5 - 6t^6 + 7t^7 - 6t^8 + 3t^9 + t^{10} - 4t^{11} + 6t^{12} - 7t^{13} + 5t^{14} - 3t^{15} + t^{16}$.

- HOMFLY polynomial: $a^{-6}z^6 + 4a^{-6}z^4 + 4a^{-6}z^2 + a^{-6} + 2a^{-8}z^6 + 9a^{-8}z^4 + 10a^{-8}z^2 + 3a^{-8} - a^{-10}z^4 - 6a^{-10}z^2 - 4a^{-10} + 2a^{-12}z^4 + 4a^{-12}z^2 + 3a^{-12} - 3a^{-14}z^2 - 3a^{-14} + a^{-16}$

- Conway polynomial: $\nabla = 1 + 9z^2 + 14z^4 + 3z^6$

As in the previous example, a quick check tells us that this diagram (identified by KnotFinder as 15_n11445 [3]) cannot represent a positive knot:

$$\max\deg V = 16 \not\leq 15 = 4\min\deg V + \text{lead coeff}\,\nabla,$$

so this fails the test of Theorem 4.2.17.

We can use this knot to generate an infinite family of knots that can be shown not to be positive by using Theorem 4.2.17 in the same way. We create a new diagram D_w by adding w copies of a three-crossing loop to the upper left corner of our original diagram D_0, as shown in Figure 4.25, so that $c(D_w) = c(D_0) + 3w$.

Claim 4.6.12. Let $V(D_w)$ be the Jones polynomial of D_w, and let $\nabla(D_w)$ be its Conway polynomial.

(a) $\min\deg V(D_w) = \min\deg V(D_0) + w$

(b) $\max\deg V(D_w) = \max\deg V(D_0) + 4w$

(c) The second Jones coefficient of $V(D_w)$ = the second Jones coefficient of $V(D_0)$

(d) $\text{lead coeff}\,\nabla(D_w) = \text{lead coeff}\,\nabla(D_0)$.

If our claims are true, then the second Jones coefficient of D_w is -2, and yet

$$\begin{aligned}\max\deg V(D_w) &= \max\deg V(D_0) + 4w \\ &= 16 + 4w \\ &\nleq 15 + 4w \\ &= 4\min\deg V(D_0) + \operatorname{lead\,coeff} \nabla(D_0) + 4w \\ &= 4\min\deg V(D_w) + \operatorname{lead\,coeff} \nabla(D_w).\end{aligned}$$

so by Theorem 4.2.17, D_w cannot be a diagram of a positive knot.

Proof. The proof is the same as for Claim 4.6.2. □

Bibliography

[1] Benjamin A. Burton, Ryan Budney, William Pettersson, et al., *Regina: Software for low-dimensional topology.*

[2] Lizzie Buchanan, *A new condition on the Jones polynomial of a fibered positive link*, 2022.

[3] Jae Choon Cha and C.-M. Michael Wong, *Knotfinder*, URL: `https://knotinfo.math.indiana.edu/homelinks/knotfinder.php`, March 2023.

[4] P. R. Cromwell, *Homogeneous Links*, Journal of the London Mathematical Society **s2-39** (1989), no. 3, 535–552.

[5] Peter Feller, Lukas Lewark, and Andrew Lobb, *Almost positive links are strongly quasipositive*, Mathematische Annalen **385** (2022), no. 1-2, 481–510.

[6] D.; Hoste J.; Lickorish W. B. R.; Millett K.; Ocneanu A. Freyd, P.; Yetter, *A new polynomial invariant of knots and links*, Bulletin (New series) of AMS **12** (1985), no. 2, 239–246.

[7] David Futer, *Fiber detection for state surfaces*, Algebraic & Geometric Topology **13** (2013), no. 5, 2799–2807.

[8] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Mathematics **2069** (2013).

[9] Louis H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), no. 3, 395–407.

[10] Tetsuya Ito and Alexander Stoimenow, *Invariants of weakly successively almost positive links*, 2022.

[11] Vaughan F. R. Jones, *A polynomial invariant for knots via von neumann algebras*, Bulletin (New series) of AMS **12** (1985), no. 1, 103–111.

[12] Charles Livingston and Allison H. Moore, *Knotinfo: Table of knot invariants*, URL: knotinfo.math.indiana.edu, March 2022.

[13] Kyle Miller, *Knotfolio*, URL: https://kmill.github.io/knotfolio/.

[14] Jozef H. Przytycki and Kouki Taniyama, *Almost positive links have negative signature*, 2009.

[15] Józef H. Przytycki and Pawel Traczyk, *Conway algebras and skein equivalence of links*, Proceedings of the American Mathematical Society **100** (1987), no. 4, 744–748.

[16] Alexander Stoimenow, *On polynomials and surfaces of variously positive links*, Journal of European Mathematical Society **7** (2005), no. 4, 477–509.

[17] Alexander Stoimenow, *On some restrictions to the values of the Jones polynomial*, Indiana University Mathematics Journal **54** (2005), 557–574.

[18] ———, *Minimal genus and fibering of canonical surfaces via disk decomposition*, Lms Journal of Computation and Mathematics **17** (2014), 77–108.

[19] Alexander Stoimenow, *Lists of non-alternating positive knots of crossing number* $\leq$ 15, URL: http://stoimenov.net/stoimeno/homepage/ptab/index.html, 2022.

[20] Keiji Tagami, *The rasmussen invariant, four-genus, and three-genus of an almost positive knot are equal*, Canadian Mathematical Bulletin **57** (2014), no. 2, 431–438.