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# Metastability in Two Dimensions and the Effective Potential

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## Abstract

We study analytically and numerically the decay of a metastable phase in (2+1)-dimensional classical scalar field theory coupled to a heat bath, which is equivalent to two-dimensional Euclidean quantum field theory at zero temperature. By a numerical simulation we obtain the nucleation barrier as a function of the parameters of the potential, and compare it to the theoretical prediction from the bounce (critical bubble) calculation. We find the nucleation barrier to be accurately predicted by theory using the bounce configuration obtained from the tree-level (“classical”) effective action. Within the range of parameters probed, we found that using the bounce derived from the one-loop effective action requires an unnaturally large prefactor to match the lattice results. Deviations from the tree-level prediction are seen in the regime where loop corrections would be expected to become important.

# 1 Introduction

In the early eighteenth century, Gabriel Daniel Fahrenheit [1] noticed that pure water could be cooled well below  $32^\circ$  on his newly invented temperature scale, and still remain a liquid. It was not until 1935, however, that Becker and Döring [2] gave a quantitative nucleation-theoretic treatment of droplet formation for fluid systems. Phenomenological field theory treatments were developed by Cahn and Hilliard [3], and by Langer [4] within the context of a coarse-grained Ginzburg-Landau model. In quantum field theory, the study of metastable vacuum decay was initiated by Voloshin, Kobzarev, and Okun [5], and put onto firm theoretical ground by Coleman and Callan [6]. The realm of applicability of homogeneous nucleation theory is extremely wide, from vapors, liquids, and solutions to metals, polymers, and glasses [7]. Recently, the possibility that first-order phase transitions occurred in the early Universe has generated a great deal of interest in the metastable decay of the vacuum itself. Well known examples are inflation [8], the electroweak phase transition [9], and the quark-hadron phase transition [10].

Surprisingly, however, classical nucleation theory has yet to receive clear experimental verification [11]: most experimental systems have complicated features that are ignored when formulating a Ginzburg-Landau description to which the classical theory could be applied. Furthermore, even when a reliable field-theoretic description is known, it is unclear how to accurately obtain the coarse-grained free-energy functional that is required for the critical-bubble calculation [12]. In cosmological applications, it has been customary to use the one-loop effective potential in the calculation of the nucleation barrier [8, 13]. However, this procedure has been questioned, since  $V_{1\text{-loop}}$  only reflects static properties of the theory, and hence may not be a good guide to the dynamics of out-of-equilibrium processes [14]. Also, the incorporation of temperature corrections to the nucleation barrier has not been carefully considered [15].

Given the universality of the topic it would seem desirable to test the classical theory within its area of validity by performing a numerical simulation of some simple model, for which we can measure the effective nucleation barrier. This barrier would then be compared with theoretical predictions. In recent work [17] we carried out this programme for a one-dimensional Ginzburg-Landau model, and showed that classical nucleation theory using the microscopic (lattice) Hamiltonian agrees well with numerical simulations. In this letter we will analyze the more realistic two-dimensional case, for which the lattice Hamiltonian is cutoff-dependent, and hence it cannot be used to determine the nucleation barrier.

The system we study is classical thermal  $(2+1)$ -dimensional scalar field theory, which is closely related to zero-temperature quantum scalar field theory in two-dimensional

Euclidean spacetime. Replacing the temperature  $T$  with the quantum of action  $\hbar$ , the equilibrium correlation functions of the classical theory map exactly onto the Green's functions of the quantum theory, and the free energy maps onto the effective action. Since we will find it convenient to mix the classical and quantum terminologies, we give a dictionary in Table 1. Even though the classical nucleation rate in Langevin time seems like a non-equilibrium quantity with no quantum analogue, classical nucleation theory identifies it (up to a dynamical factor) with the imaginary part of the equilibrium free energy density of the metastable state [4]. Under  $T \rightarrow \hbar$  this maps onto the quantum tunnelling rate, which is given in terms of the imaginary part of the energy density of the false vacuum [6]. Therefore, one could also think of our simulation as an independent way of calculating the barrier to zero temperature quantum tunnelling. In the classical language, we will show that the barrier obtained from the tree-level free energy (the Hamiltonian), ignoring entropic (loop) corrections, is in excellent agreement with the numerical results. In the quantum language, what we will show is that the tree-level (classical) action yields the correct bounce action in the Callan and Coleman calculation. Naturally this result may be modified when the scalar field interacts with other fields [22, 15].

There has been only one previous numerical study of nucleation, that of Valls and Mazenko [16], which found that the theory gave a very poor estimate of the nucleation barrier in a two-dimensional Ginzburg-Landau model coupled to a heat bath. There are several reasons [17] why our results should differ from theirs, the dominant one being that two-dimensional classical thermal field theory suffers from ultra-violet divergences which produce severe lattice-spacing dependence of many measured quantities, including nucleation rates. Continuum results can be obtained, as we will show, by introducing cutoff-dependent counterterms and renormalizing. Since Valls and Mazenko did not explicitly introduce counterterms, it is possible that the continuum theory they were simulating was different from the one for which they were calculating theoretical decay rates.

We will now review the “critical bubble” or “bounce” method of calculating the decay rate, before proceeding to test it numerically.

## 2 The Theoretical Results

Consider a classical (2+1)-dimensional scalar field theory with an asymmetric double-well potential, at finite temperature. By virtue of its relation to a two-dimensional quantum field theory, mentioned above, we expect this system to be formally ill-defined because

Classical Statistical Mechanics	Euclidean Quantum Field Theory
Microscopic (Lattice) Hamiltonian $H_{\text{latt}} = \int \mathcal{H}_{\text{latt}} d^2x$	Bare Action $S_{\text{bare}} = \int \mathcal{L}_{\text{bare}} d^2x$
Lattice Energy Density $\mathcal{H}_{\text{latt}} = \frac{1}{2}Z_{\text{latt}} \nabla\phi ^2 + V_{\text{latt}}(\phi; a)$	Bare Lagrangian Density $\mathcal{L}_{\text{bare}} = \frac{1}{2}Z_{\text{bare}} \nabla\phi ^2 + V_{\text{bare}}(\phi; a)$
Free Energy $F = \int \mathcal{F} d^2x$	Effective Action $S_{\text{eff}} = \int \mathcal{L}_{\text{eff}} d^2x$
Free Energy Density $\mathcal{F} = \frac{1}{2}Z_F \nabla\phi ^2 + V_F(\phi)$	Effective Lagrangian Density $\mathcal{L}_{\text{eff}} = \frac{1}{2}Z_{\text{eff}} \nabla\phi ^2 + V_{\text{eff}}(\phi)$
$F(\phi, T) = \underbrace{H(\phi)}_{\text{Hamiltonian}} + \underbrace{\sum_{n=1}^{\infty} \Delta F_{n\text{-loop}}(\phi) T^n}_{\text{Entropic corrections}}$ $V_F(\phi, T) = U(\phi) + \sum_{n=1}^{\infty} \Delta V_{n\text{-loop}}(\phi) T^n$	$S_{\text{eff}}(\phi, \hbar) = \underbrace{S_{\text{class}}(\phi)}_{\text{Classical action}} + \underbrace{\sum_{n=1}^{\infty} \Delta S_{n\text{-loop}}(\phi) \hbar^n}_{\text{Quantum corrections}}$ $V_{\text{eff}}(\phi, \hbar) = \underbrace{U(\phi)}_{\text{Classical potential}} + \sum_{n=1}^{\infty} \Delta V_{n\text{-loop}}(\phi) \hbar^n$

Table 1: Dictionary of equivalent terms for classical thermal and quantum mechanical two-dimensional field theory on a lattice with spacing  $a$ .

of ultra-violet divergences in the correlation functions. For the moment we therefore imagine integrating out the high-momentum physics, and assume that the low-momentum dynamics is described by some coarse-grained free-energy functional,

$$\frac{1}{T}F_{\text{dyn}}[\phi] = \frac{1}{T} \int \left\{ \frac{1}{2}Z_{\text{dyn}}|\nabla\phi|^2 + V_{\text{dyn}}(\phi) \right\} d^2x \quad . \quad (2.1)$$

We assume that  $V_{\text{dyn}}(\phi)$  has a local minimum at  $\phi = 0$ , separated by a barrier from a lower global minimum at some positive (perhaps infinite)  $\phi$ .

In general one does not know what  $F_{\text{dyn}}$  should be. Langer [12] states that it should be a free energy coarse-grained up to the correlation length (but no further). We will investigate two candidates for  $F_{\text{dyn}}$ , the tree-level free energy or Hamiltonian  $H$  (the classical action, in the quantum language) and the 1-loop free energy (1-loop effective action) (see Table 1).

We will study nucleation for potentials of the general form

$$U(\phi) = \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\phi^3 + \frac{1}{24}h\phi^4 \quad . \quad (2.2)$$

This seems to give a 4-parameter family of theories, but in fact by rescaling the field and distance, we can reduce it to a functional involving only two (dimensionless) parameters,  $\lambda$  and  $\theta$ :

$$\frac{1}{T}H[\phi] = \frac{1}{\theta} \int \left\{ \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} \phi^2 - \frac{1}{6} \phi^3 + \frac{1}{24} \lambda \phi^4 \right\} d^2x \quad , \quad (2.3)$$

where

$$\theta = \frac{g^2 T}{m^4} \quad \text{and} \quad \lambda = \frac{\hbar m^2}{g^2} \quad (2.4)$$

The dimensionless parameter  $\theta$  plays the role of temperature, while  $\lambda$  parameterizes the different potentials. The potential has one minimum at  $\phi = 0$ , and for  $0 < \lambda < \frac{1}{3}$  there is a second, lower minimum at positive  $\phi$ , separated from the metastable minimum by a maximum at  $\phi_{\max} = (3/2\lambda)(1 - \sqrt{1 - 8\lambda/3})$ .

For such a system, we expect  $F_{\text{dyn}}$  to be a functional parameterized by  $\lambda$  and  $\theta$ , such that if the system starts out with thermal expectation  $\langle \phi(\vec{x}) \rangle = 0$  at finite temperature  $\theta$  then it will make a thermally activated transition to the global minimum with a characteristic rate per unit area

$$\Gamma = A \exp(-B/\theta) \quad , \quad (2.5)$$

where  $A$ , the prefactor, is a function of  $\lambda$ ,  $\theta$ , and the coupling to the heat bath  $\eta$  (see Sect. 3).  $B$  is the nucleation barrier, which, according to the classical nucleation or bounce calculation [4, 6], is the energy of the “critical bubble” or “bounce” saddle point of  $F_{\text{dyn}}$ . The critical bubble configuration  $\phi_B(\vec{x})$  is given by the static (circularly symmetric) solution to the equation of motion,

$$\frac{\partial^2 \phi_B}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_B}{\partial r} = Z_{\text{dyn}}^{-1} \frac{\partial V_{\text{dyn}}}{\partial \phi}(\phi_B) \quad , \quad (2.6)$$

with the boundary conditions that the field be regular at the origin ( $\phi'_B(0) = 0$ ) and in the false vacuum at spatial infinity ( $\phi_B(\infty) = 0$ ). If one thinks of  $r$  as being “time”, this equation describes the motion of a particle with position  $\phi_B$  in the potential  $-V_{\text{dyn}}(\phi)$ , with a frictional resistance that is inversely proportional to the time. In general there is no closed-form solution to Eq. (2.6), but it can be solved numerically by guessing a value for  $\phi_B(0)$  and evolving forward in  $r$ . If the solution overshoots and  $\phi_B$  becomes negative then the initial value was too high; if  $\partial \phi_B / \partial r$  becomes positive then it was too low. In this way we converge on the correct solution, and its energy is the nucleation barrier  $B$ . Since in general  $F_{\text{dyn}}$  may depend on  $\theta$ , we may write  $B = B_0 + B_1 \theta + B_2 \theta^2 + \dots$ , and so Eq. (2.5) becomes

$$\Gamma = A \exp(-B_1) \exp[-(B_0/\theta + B_2 \theta + \dots)] \quad . \quad (2.7)$$

As was mentioned above, the tree-level candidate for  $F_{\text{dyn}}$  is the Hamiltonian  $H$ . This is independent of  $\theta$ , so  $B$  will just be  $B_0$ , and we can calculate it by the method described above, setting  $V_{\text{dyn}} = U$ . Another possibility is that  $F_{\text{dyn}}$  is  $H$  plus some entropic corrections. For example, in studies of cosmological phase transitions it is generally assumed to be the real part of the 1-loop free energy. In this case  $B_1$ ,  $B_2$  etc are non-zero, but as  $\theta \rightarrow 0$  the entropic corrections disappear, so  $B_0$  is still given by the tree-level ( $F_{\text{dyn}} = H$ ) barrier prediction. The lowest order correction  $B_1$  only modifies the prefactor, not the measured barrier. Thus entropic corrections will primarily be visible through the effects of the  $B_2$  term, since we have little information about the theoretical value of  $A$  beyond the fact that we expect it to be of order 1 on dimensional grounds. It will turn out that we can confirm that  $B_0$  is given by the tree-level bounce, but our observations cover too narrow a range of  $\theta$  for the  $B_2$  predicted by the 1-loop free energy to be visible, so we are unable to say definitively whether it gives the right entropic corrections.

We expect the barrier to go to infinity when the two minima are degenerate, at  $\lambda = \frac{1}{3}$ , and to go to some finite value as  $\lambda \rightarrow 0$ . In fact,  $\lambda$  can take on negative values and everything still works exactly as described above. There is no longer a global minimum to the potential, but there is still a metastable state which decays by surmounting a barrier that can be calculated by finding the critical bubble. We will restrict ourselves to studying  $\lambda$  in the range 0 to 0.32, since negative  $\lambda$  is not physically relevant, and as  $\lambda \rightarrow \frac{1}{3}$  the barrier becomes large, requiring high temperatures to induce nucleation, which invalidates the loop expansion we use in the next section.

### 3 The Lattice Formulation

In order to test the critical bubble theory, we need to simulate the classical dynamics of the (2+1)-dimensional scalar field theory in contact with a heat bath at temperature  $\theta$ . This may be done by evolution of the stochastic Langevin equation,

$$\frac{\partial^2 \phi}{\partial t^2} = Z_{\text{latt}} \nabla^2 \phi - \eta \frac{\partial \phi}{\partial t} - \frac{\partial V_{\text{latt}}}{\partial \phi} + \xi(x, t) \quad , \quad (3.1)$$

where  $\eta$  is the viscosity coefficient, and  $\xi$  is the stochastic noise with vanishing mean, related to  $\eta$  by the fluctuation-dissipation theorem,

$$\langle \xi(\vec{x}, t) \xi(\vec{x}', t') \rangle = 2\eta\theta \delta(t - t') \delta^2(\vec{x} - \vec{x}') \quad . \quad (3.2)$$

In writing Eqns. (3.1) and (3.2), we implicitly assumed that the system is Markovian, *i.e.*, the correlation time scale for the noise is much smaller than the typical relaxation time

for the system, which is the inverse of the oscillation frequency around the metastable equilibrium.

In principle it now seems straightforward to put the field theory on a two-dimensional lattice, and solve Eq. (3.1) numerically. However, (3.1) is expressed in terms of the lattice (bare) parameters, so in order to make contact with the calculation of Sect. 2 we must explicitly construct a lattice potential that has a definite continuum limit (lattice spacing  $a \rightarrow 0$ ), and have some way of specifying which continuum theory it corresponds to. Since the theory has exactly the same divergences as a two-dimensional Euclidean quantum field theory, the lattice potential will have to contain lattice-spacing-dependent counterterms in order to give a good continuum limit. (For a review, see Parisi [18], Ch. 5).

Our strategy will be to use continuum field theory with a hard momentum cutoff  $\Lambda \sim a^{-1}$  as our guide. We can easily calculate the free energy (effective action) for such a theory with a given bare potential by a loop expansion, which is an expansion in powers of  $\theta$  ( $\hbar$  in the quantum theory). This will tell us what counterterms are needed to cancel the cutoff dependence of the lattice theory. We will only have to calculate to one loop, since there are no divergent graphs beyond one loop.

We studied the lattice action defined by (see Table 1):

$$\begin{aligned} Z_{\text{latt}} &= 1 \quad , \\ V_{\text{latt}}(\phi) &= U(\phi) + V_{\text{ct}}(\phi) \quad , \\ U(\phi) &= \frac{1}{2}\phi^2 - \frac{1}{6}\phi^3 + \frac{1}{24}\lambda\phi^4 \quad , \\ V_{\text{ct}}(\phi) &= -\frac{\theta}{4\pi} \ln(aM_1(\lambda, \theta))\phi + \frac{\lambda\theta}{8\pi} \ln(aM_2(\lambda, \theta))\phi^2 \quad . \end{aligned} \tag{3.3}$$

By the standard calculation [19] for a scalar quantum field theory in two Euclidean dimensions with classical potential  $U(\phi)$  and ultraviolet cutoff  $\Lambda$ , we find that the 1-loop effective potential is

$$V_F = U + V_{\text{ct}} + \frac{\theta}{2} \int_0^\Lambda \frac{d^2k}{(2\pi)^2} \ln\left(1 + \frac{U''}{k^2}\right) \quad . \tag{3.4}$$

Performing the integral and dropping terms independent of  $\phi$ , we find for large  $\Lambda$ ,

$$V_F = U + V_{\text{ct}} + \frac{\theta}{8\pi} \left( U'' \ln(\Lambda^2) + U'' - U'' \ln(U'') \right) \quad . \tag{3.5}$$

The counterterms cancel the  $\Lambda$ -dependence, so we expect the theory with lattice potential (3.3) to have a good continuum limit. What effective potential will this continuum theory have? Eq. (3.5) was calculated for a theory with momentum cutoff  $\Lambda$  to simulate the effects of the lattice. It does not tell us what finite parts will be left from the cancellation



of the cutoff-dependence in the actual lattice theory. Thus we can only say that we expect the lattice theory defined by (3.3) to have one-loop effective potential

$$V_F = U(\phi) + \frac{\theta}{8\pi} \left( U''(\phi) - U''(\phi) \ln(U''(\phi)) \right) + f(M_1)\phi + g(M_2)\phi^2 \quad . \quad (3.6)$$

$M_1$  and  $M_2$  are to be fixed by a renormalization condition, but since the functions  $f$  and  $g$  are unknown, this will have to be done numerically for the lattice theory. Our renormalization condition was

$$f(M_1) = g(M_2) = 0 \quad . \quad (3.7)$$

It only remains to calculate the 1-loop effective field normalization  $Z$  (see Table 1). It is given in terms of the one-particle-irreducible Green function  $\Gamma^{(2)}(p)$ , and is cutoff-independent in two dimensions. Evaluating the relevant Feynman diagram,

$$Z = \frac{\partial \Gamma^{(2)}}{\partial p^2}(0) = 1 - \frac{\theta}{48\pi} + \mathcal{O}(\theta^2) \quad (3.8)$$

To summarize, the lattice theory (3.3) with renormalization conditions (3.7) will have one-loop free energy density (effective action)

$$\mathcal{F} = \left( 1 - \frac{\theta}{48\pi} \right) |\nabla\phi|^2 + U(\phi) + \frac{\theta}{8\pi} \left( U''(\phi) - U''(\phi) \ln(U''(\phi)) \right) \quad (3.9)$$

$U''$  is negative between the points of inflection of  $U$ , so  $\mathcal{F}$  is complex. Weinberg and Wu [21] have suggested that the real part of this is physically meaningful, so when using  $\mathcal{F}$  in bounce calculations, we take the real part.

All these calculations are valid to one loop. Without explicitly evaluating the higher corrections, we can estimate that they will be valid for  $\theta \ll 8\pi$ , at least near the minima of the potential. There are no additional divergences at two loops, but finite corrections to  $\mathcal{F}$ , proportional to  $\theta^2$ , will appear. This puts a limit on how close we can push  $\lambda$  to the value  $1/3$ , since the barrier becomes large as the two minima in  $U(\phi)$  become degenerate, requiring high temperatures in order to observe nucleations on the lattice.

## 4 The Numerical Analysis and Results

We now have a way to test the theoretical calculation of Sect. 2. For each value of  $\lambda$  we measure the nucleation barrier on the lattice by measuring nucleation rates for a range of values of temperature  $\theta$ . For each  $\lambda$  and  $\theta$  this involves two stages. Firstly, we ensure that we are looking at the right continuum theory, by running simulations with lattice

potential (3.3), and varying  $M_1$  and  $M_2$  until (3.7) is obeyed. We then know that we have a theory whose free energy is given by (3.9), to first order in  $\theta/8\pi$ . The second stage involves measuring the nucleation rate for this theory. In both stages we ensure that we are studying the continuum limit by reducing the lattice spacing  $a$  until the results become independent of  $a$ . We checked the dependence of our results on the lattice length, the time step, the random number generator and the random number seed. Within the limits of our numerical accuracy we found that the lattice approximation correctly describes the continuum field theory. We chose  $\eta = 1$  as the viscosity in all simulations. In principle, knowledge of the physical nature of the heat bath would enable one to calculate the value of  $\eta$ , as well as additional nonlinear or nonlocal dissipative terms in the Langevin equation.

The easiest way to impose the renormalization conditions was to choose  $M_1$  and  $M_2$  such that the measured true and false vacua on the lattice coincided with the minima of the free energy (3.9). This required running a simulation of (3.1) on a relatively small lattice ( $L = 10$ ), with initial conditions that caused the field to settle down in the appropriate vacuum state. It was then a simple matter to measure the thermal expectation value of the field. We found that the renormalization condition  $f(M_1) = 0$  (*i.e.*  $\langle\phi\rangle = 0$  in the metastable state) was obeyed for  $M_1 = 1/(2\pi)$ , in the sense that with this choice the measured metastable average field was always much less than the peak  $\phi_{\max}$ . The condition  $g(M_2) = 0$  was obeyed for  $M_2 = 1/(2\pi)$ , in the sense that the measured true vacuum was within a few percent of the value predicted by (3.9). Obviously this method could not fix  $M_2$  in the  $\lambda = 0$  theory, since the true vacuum is then at infinite  $\phi$ . However in that case there is no cutoff-dependent  $\phi^2$  term in the free energy, and  $g(M_2) = 0$ .

To measure the nucleation rates, we evolved the Langevin equation on square lattices of size  $L = 20, 40, 60$  or  $120$ , with lattice spacing  $a = 0.5$ , using a leapfrog algorithm with time step of  $0.05$ . For each  $\lambda$  and  $\theta$  we performed several hundred simulations. In each one we started the system in the metastable state, evolved the Langevin equation (3.1) forward in time, measuring the time  $t$  that elapsed before it escaped from the metastable region by nucleating a growing bubble of the stable state. From this data we constructed a frequency histogram for nucleation times from which  $\Gamma(\lambda, \theta)$ , the nucleation rate per unit area, could be read off by fitting to the expected form:  $\text{Prob}(t) \propto \exp(-t\Gamma L^2)$  for a lattice of size  $L$ . At low temperatures  $\Gamma$  gets small (2.5), but we could compensate for this by increasing  $L$ . Thus with a range of lattice sizes we were able to observe a wider range of temperatures than would have been possible using only one lattice. We were still constrained to a fairly narrow temperature range, since the simulation ran much more

slowly on larger lattices.

In order to determine when nucleation had occurred, we calculated a smoothed field by averaging  $\phi$  over square blocks of size  $\Delta L$ . The system was considered to have escaped from metastability when one or more of the blocks achieved an average field value greater than the peak  $\phi_{\max}$  (see Sect. 2). For each value of  $\lambda$  we chose the smallest  $\Delta L$  such that when one block was converted, the rest of the lattice would always follow. For  $\lambda = 0, 0.1, 0.2$  we set  $\Delta L = 5$ ; For  $\lambda = 0.25, 0.3, 0.31$  we set  $\Delta L = 10$ ; for  $\lambda = 0.32$  we set  $\Delta L = 20$ .

Our results are summarized in Fig. 1, 2, and 3.

In Fig. 1 we show a logarithmic plot of the inverse nucleation rate per unit area  $\Gamma^{-1}$ , as a function of inverse temperature, for different values of the potential parameter  $\lambda$ . From Eq. (2.7) for the decay rate we expect that  $\ln(\Gamma^{-1}) = \ln(A^{-1}) + B_1 + B_0(\lambda)/\theta + B_2\theta$ . The excellent straight line fits for  $\lambda \leq 0.3$  indicate that the classical theory that gave rise to Eq. (2.7) is qualitatively correct in this regime. There is no sign of a  $B_2\theta$  term. We conclude that the nucleation barrier is well approximated by the form  $B = B_0 + B_1\theta$  for the ranges of temperature probed for  $\lambda \leq 0.3$ . For  $\lambda = 0.31, 0.32$  there are signs of a  $B_2$  term, so entropic corrections to the barrier are becoming important, as one would expect at high  $\theta$ .

In Fig. 2 we confront the numerical barrier measurements from Fig. 1 with the theoretical prediction for the nucleation barrier  $B_0$  given in Sect. 2, which involves taking  $V_{\text{dyn}}(\phi)$  to be  $U(\phi)$ , the potential appearing in the tree-level free energy (Hamiltonian, or classical action). For  $\lambda \leq 0.3$  we obtain almost perfect quantitative agreement between theory and numerical experiment, confirming that  $B_0$  is accurately given by the action of the tree-level bounce. As one might expect from the weakness of their straight line fits in Fig. 1, the barriers for  $\lambda = 0.31, 0.32$  are not well predicted by the tree-level bounce.

In Fig. 3 we plot the values of the prefactor  $A$  that would fit the data of Fig. 1, using the tree-level bounce and the 1-loop bounce. For the tree-level points, we fitted Eq. (2.7) to the data of Fig. 1 with  $B_1 = 0$ , and plotted the value of  $A^{-1}$  for each value of  $\lambda$ . For the 1-loop points we calculated  $B(\lambda, \theta)$  using the method of Sect. 2 with  $F_{\text{dyn}}$  set to the real part of the 1-loop free energy (3.9). (This was only possible for  $\theta \lesssim 10$ . For larger  $\theta$  no solution could be found, so 1-loop barrier predictions could not be given for  $\lambda = 0.31, 0.32$ , whose rate measurements were all at larger  $\theta$ .) By fitting to the form  $B = B_0(\lambda) + B_1(\lambda)\theta$  we obtained  $B_0$  (which agreed with the tree-level prediction, as expected) and  $B_1$ . For each  $\lambda$  we then found  $A$  by fitting Eq. (2.7) to the data of Fig. 1.

Much larger prefactors were required to fit the 1-loop predictions to the data than for the tree-level predictions. Since  $A$  is expected to be of order 1, this provides

circumstantial evidence that the one-loop bounce is not the relevant quantity. For a definitive answer, a more powerful computer would be needed, allowing a larger range of  $\theta$  to be probed for each  $\lambda$ , so that the value of  $B_2$  could be measured.

The fact that the results fit the theory less well as  $\lambda$  approaches  $\frac{1}{3}$  should not be surprising. Since the barrier is diverging in this region, we are forced to use high  $\theta$  in order to see any nucleations, and so we expect entropic corrections to the barrier to become important. The deviations begin at  $\lambda = 0.31$ , where temperatures in the range 12 to 20 are investigated. This is approaching the regime  $\theta \gtrsim 8\pi$ , where the loop approximation breaks down, so we expect significant  $\theta$ -dependent corrections to the theoretical nucleation barrier. In other words, we expect  $B_2$  and higher terms to become large. It is perhaps more surprising that we do not see deviations for  $\lambda = 0.3$ , where a similar temperature range was explored.

## 5 Conclusions

We have shown that, for a single scalar field in two space dimensions, classical nucleation theory correctly predicts the nucleation barrier to within a few percent, when the Hamiltonian, *i.e.* the tree-level free energy, is used in the calculation. Equivalently (see Table 1), the quantum bounce calculation correctly predicts the barrier, as long as the tree-level (“classical”) action is used.

Obviously this statement only makes sense for the parameter range where a loop expansion is possible, and, as expected on the basis of the calculations of Sect. 3, we found that it broke down for large loop parameter,  $\theta/8\pi \gtrsim 1$ . The interesting thing is that the tree-level action gave correct predictions for  $\theta/8\pi$  as large as 0.5, a regime where one might have expected significant loop corrections.

We were unable to make any definitive statement about the 1-loop bounce (the bounce solution obtained by using the 1-loop effective potential). Fig. 3 shows that it requires an unnaturally large prefactor, but the temperature range we probed was too narrow to see if its predicted higher-order corrections were present. With the aid of more powerful computing resources it should be possible to settle this question.

There are many other directions in which this work can be extended. E. Weinberg [22] has recently argued that a truncated 1-loop effective potential correctly describes nucleation in certain radiative symmetry breaking models, and it would be interesting to study this numerically. Another issue is finding the dependence of the prefactor not only on the temperature but also on the coupling to the heat bath. We have previously studied the dependence of the thermal kink-antikink pair creation on the coupling to

the bath [20], finding good agreement with Kramer's celebrated result [23]. Perhaps of more relevance to current work on out-of-equilibrium processes in the early Universe, we could study the dynamics of weakly first-order transitions. Recent work has claimed that for sufficiently weak transitions the usual bubble nucleation mechanism discussed here must be modified. Instead, it has been suggested that the transition may evolve through the nucleation and subsequent percolation of sub-critical bubbles in such a way as to resemble an emulsion of phases as the system is cooled below the critical temperature [24]. We are presently investigating these questions.

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## Figure Captions

Figure 1: Numerically measured inverse nucleation rate  $\Gamma^{-1}$  as a function of inverse temperature, with straight line fits, for a range of values of the dimensionless  $\phi^4$  coupling  $\lambda$ . From the right:  $\lambda = 0, 0.1, 0.2, 0.25, 0.3, 0.31, 0.32$ .

Figure 2: Nucleation barrier as a function of  $\lambda$ . The points are the slopes of the lines in Fig. 1. The line is the critical bubble calculation of Sect. 2 at tree level, *i.e.* with  $Z_{\text{dyn}} = 1$  and  $V_{\text{dyn}} = U$ .

Figure 3: Rate prefactor  $A^{-1}$  as a function of  $\lambda$ . Bottom set of points is for the tree-level prediction (see Eq. (2.5)) fitted to the data of Fig. 1. Top set of points is for the 1-loop prediction (see Eq. (2.7)). We excluded  $\lambda = 0.31, 0.32$  because their error bars were too large.