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According to classical theories, the frequencies of internal-gravity waves in stratified rotating fluids must lie between the Brunt-Väisälä frequency (a measure of the vertical density stratification) and the Coriolis frequency (equal to twice the rotation rate about the vertical axis). It is shown here that, in the case of the Earth's rotation where the pole-to-pole axis of rotation is almost everywhere not parallel to the local vertical, the range of realizable frequencies is broader. New formulas are derived for the lower and upper bounds of the frequencies. [S0031-9007(96)01859-5]

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The classical theory of internal-gravity waves in a stably stratified fluid [1–3] yields a dispersion relation that constrains the wave frequency not to exceed the so-called Brunt-Väisälä frequency, a direct measure of the fluid's density stratification supporting the waves. Application of this theory to geophysical fluids such as the atmosphere or the ocean requires modification to incorporate the effect of the Earth's rotation. Traditionally, the insertion of the Coriolis acceleration is limited to one new term in each of the horizontal-momentum equations [3–5], and the dispersion relation is augmented by a term depending on the so-called Coriolis frequency, equal to twice the projection of the rotation vector onto the vertical direction. An elementary analysis of this dispersion relation yields that the wave frequency is now constrained to lie between the Brunt-Väisälä frequency and the Coriolis frequency. Since in most geophysical fluids the former is larger than the latter, the Brunt-Väisälä frequency acts as an upper bound, and the Coriolis frequency as a lower bound. Almost everywhere on the Earth, however, the pole-to-pole rotation axis is not parallel to the local vertical, and the Coriolis acceleration is not limited to the horizontal plane. This casts doubt on the traditional conclusion regarding the range of internal-wave frequencies in geophysical fluids. The purpose of this Letter is to reformulate the theory of internal-gravity waves in the presence of the full, three-dimensional Coriolis acceleration and thereby to derive new bounds on the range of realizable frequencies.

Departing from the traditional theory of internal-gravity waves only by retaining the full, three-dimensional Coriolis acceleration, we state the linearized equations governing the three-dimensional flow associated with a small perturbation to a basic state of motionless and hydrostatic density stratification [3],

$$\frac{\partial u}{\partial t} - fv + f_* w = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{1a}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{1b}$$

$$\frac{\partial w}{\partial t} - f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b, \tag{1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1d}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0, \tag{1e}$$

where the variables are u, v, and w, the velocity components along the x (eastward), y (northward), and z (upward) Cartesian directions, p, the pressure anomaly (equal to the total pressure minus the hydrostatic pressure of the basic state), and b, the buoyancy anomaly (equal to minus the gravitational acceleration g times the relative density anomaly caused by the motion). The coefficients are the two Coriolis frequencies, defined as

$$f = 2\Omega \sin\varphi, \qquad f_* = 2\Omega\cos\varphi, \tag{2}$$

where Ω is the rotation rate of the Earth (nearly $2\pi/24$ hr) and φ is the local latitude, a reference density ρ_0, and the square of the Brunt-Väisälä frequency defined by

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho(z)}{dz}, \quad N > 0, \tag{3}$$

where $\rho_0 + \rho(z)$ is the density profile of the basic state. Assuming a limited range of latitudes and a uniform vertical stratification, we may take all coefficients as constant and seek wave solutions of the form $\exp[i(lx + my + nz - \omega t)]$. Some algebra yields the following dispersion relation between the frequency ω and the wave number components l, m, and n:

$$\omega^2 = \frac{N^2(l^2 + m^2) + (fn + f_* m)^2}{l^2 + m^2 + n^2}. \tag{4}$$

Note that the grouping $fn + f_* m$ represents the projection of the wave number onto the rotation axis.

To determine the range of realizable frequencies, we seek the extremal values of ω as l, m, and n assume any real value between $-\infty$ and $+\infty$. While this may be done immediately, it is instructive to consider first the particular cases of existing theories.

The simplest of all cases is the one with no rotation ($f = f_* = 0$) and under the assumption of hydrostatic balance [no $\partial w/\partial t$ term on the left-hand side of (1c)]. The dispersion relation is then reduced to

$$\omega^2 = N^2\frac{l^2 + m^2}{n^2}, \tag{5}$$

and the frequency ω can take any value between zero and infinity. Reinstating the vertical acceleration $\partial w/\partial t$ in the vertical-momentum equation (1c), we obtain the corrected dispersion relation [1–3],

$$\omega^2 = N^2 \frac{l^2 + m^2}{l^2 + m^2 + n^2}, \tag{6}$$

which now limits ω^2 to N^2 and thus places an upper bound on the frequency. [Note that higher frequencies than N are permitted if the waves are evanescent in the vertical ($n^2 < 0$), but such waves are no longer called internal-gravity waves.]

If we further retain the Coriolis terms due to the vertical component of the rotation vector ($f \neq 0, f_* = 0$), we obtain the traditional dispersion relation for internal gravity waves in a stratified rotating fluid [4,5],

$$\omega^2 = N^2 \frac{l^2 + m^2}{l^2 + m^2 + n^2} + f^2 \frac{n^2}{l^2 + m^2 + n^2}. \tag{7}$$

In this dispersion relation, ω^2 appears as a weighted average between N^2 and f^2, and must therefore lie between these values. Since in geophysical flows, the Coriolis frequency is typically much smaller than the Brunt-Väisälä frequency, f acts as a lower bound (in magnitude since f may be negative), and N as an upper bound. In the hydrostatic limit [neglect of $\partial w/\partial t$ in (1c) again], the denominators $l^2 + m^2 + n^2$ reduce to n^2, and the dispersion relation becomes [6]

$$\omega^2 = N^2 \frac{l^2 + m^2}{n^2} + f^2, \tag{8}$$

in which N no longer acts as an upper bound but f still acts as a lower bound. In sum, the traditional theory concludes that the ambient rotation sets a lower bound on the frequency of internal gravity waves (namely, the Coriolis frequency, equal to twice the projection of the rotation vector onto the local vertical), and that the vertical acceleration sets an upper bound (namely, the Brunt-Väisälä frequency).

According to the generalized dispersion relation, (4), the traditional theory leading to (7) is inconsistent. Either the horizontal wave number $(l^2 + m^2)^{1/2}$ is negligible compared to the vertical wave number n, or it is not. If it is, then (7) is not better than (8), and inclusion of the vertical acceleration is unnecessary. If it is not, then $f_* m$ is most likely [7] comparable to fn in the numerator of (4), and not only the vertical acceleration but also the Coriolis acceleration must be retained in the vertical-momentum equation.

We now seek the extremal values that the frequency ω can achieve under the most general dispersion relation (4). First putting to zero the derivative of ω with respect to l, we obtain

$$\text{either} \quad l = 0 \quad \text{or} \quad (fn + f_* m)^2 = N^2 n^2. \tag{9}$$

In the second case, ω^2 becomes N^2, and we conclude that N is a stationary value of ω. As we will shortly see, this will be neither a maximum nor a minimum. In the first case ($l = 0$), ω^2 reduces to

$$\omega^2 = \frac{N^2 m^2 + (fn + f_* m)^2}{m^2 + n^2} = \frac{(N^2 + f_*^2)\lambda^2 + 2ff_*\lambda + f^2}{\lambda^2 + 1}, \tag{10}$$

where $\lambda = m/n$ is the single wave number variable on which ω depends. Setting to zero the derivative of ω with respect to λ yields

$$ff_*\lambda^2 - (N^2 - f^2 + f_*^2)\lambda - ff_* = 0, \tag{11}$$

which always admits two real roots,

$$\lambda = \frac{m}{n} = \frac{1}{2ff_*}\left[(N^2 - f^2 + f_*^2) \pm \sqrt{(N^2 - f^2 + f_*^2)^2 + 4f^2 f_*^2}\right]. \tag{12}$$

The corresponding stationary values of ω^2 are

$$\omega_\pm^2 = \frac{1}{2}\left[(N^2 + f^2 + f_*^2) \pm \sqrt{(N^2 - f^2 + f_*^2)^2 + 4f^2 f_*^2}\right], \tag{13}$$

for which it is relatively easy to show that the following rankings hold:

$$0 < \omega_-^2 < N^2 < \omega_+^2 \quad \text{and} \quad 0 < \omega_-^2 < f^2 < \omega_+^2. \tag{14}$$

In conclusion, the minimum value of ω is the positive root of ω_-^2, and its maximum value is the positive root of ω_+^2, while N is an intermediate stationary value. It also follows that $\omega_{\min}$ is lower than the minimum of $|f|$ and N, and that $\omega_{\max}$ is higher than the maximum of $|f|$ and N. In other words, the range of realizable frequencies is broader than the interval between $|f|$ and N on both low and high frequency sides. For $N \gg \Omega$, asymptotic expressions are $\omega_{\min}^2 \approx f^2(1 - f_*^2/N^2)$ and $\omega_{\max}^2 \approx N^2 + f_*^2$. This may explain why internal-wave spectra derived from oceanographic data do indeed show a range of frequencies slightly broader than the interval from $|f|$ and N [8], especially at low latitudes.

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