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Kenneth P. Bogart  
*Dartmouth College*

Garth Isaak  
*Dartmouth College*

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## Proper and unit bitolerance orders and graphs

Kenneth P. Bogart<sup>\*1</sup>, Garth Isaak<sup>2</sup>

*Department of Mathematics and Computer Science, Dartmouth College, 6188 Bradley Hall, Hanover,  
NH 03755-3551, USA*

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### Abstract

We say any order  $<$  is a *tolerance order* on a set of vertices if we may assign to each vertex  $x$  an interval  $I_x$  of real numbers and a real number  $t_x$  called a tolerance in such a way that  $x < y$  if and only if the overlap of  $I_x$  and  $I_y$  is less than the minimum of  $t_x$  and  $t_y$  and the center of  $I_x$  is less than the center of  $I_y$ . An order is a *bitolerance order* if and only if there are intervals  $I_x$  and real numbers  $t_1(x)$  and  $t_r(x)$  assigned to each vertex  $x$  in such a way that  $x < y$  if and only if the overlap of  $I_x$  and  $I_y$  is less than both  $t_r(x)$  and  $t_1(y)$  and the center of  $I_x$  is less than the center of  $I_y$ . A tolerance or bitolerance order is said to be bounded if no tolerance is larger than the length of the corresponding interval. A *bounded tolerance graph* or *bitolerance graph* (also known as a *trapezoid graph*) is the incomparability graph of a bounded tolerance order or bitolerance order. Such a graph or order is called *proper* if it has a representation using intervals no one of which is a proper subset of another, and it is called *unit* if it has a representation using only unit intervals. In a recent paper, Bogart, Fishburn, Isaak and Langley (1995) gave an example of proper tolerance graphs that are not unit tolerance graphs. In this paper we show that a bitolerance graph or order is proper if and only if it is unit. For contrast, we give a new view of the construction of Bogart et al. (1995) from an order theoretic point of view, showing how linear programming may be used to help construct proper but not unit tolerance orders.

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### 1. Introduction

An order  $<$  on a set  $X$  is called an *interval order* if it is possible to assign to its vertices  $x$  intervals  $I_x$  of real numbers in such a way that  $x < y$  if and only if all members of  $I_x$  are less than all members of  $I_y$ . We will discuss only orders with  $|X|$  finite. An order is a (*min*) *tolerance (interval) order* on a set of vertices if we may assign to each vertex  $x$  an interval  $I_x$  of real numbers and a real number  $t_x$  called a tolerance in such a way that  $x < y$  if and only if the overlap of  $I_x$  and  $I_y$  is less than the minimum of  $t_x$  and  $t_y$  and the center of  $I_x$  is less than the center of  $I_y$ . An order is a (*min*)

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\*Corresponding author. E-mail: k.p.bogart@dartmouth.edu.

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*bitolerance (interval) order* if and only if there are intervals  $I_x$  and real numbers  $t_l(x)$  and  $t_r(x)$  assigned to each vertex  $x$  in such a way that  $x < y$  if and only if the overlap of  $I_x$  and  $I_y$  is less than both  $t_r(x)$  and  $t_l(y)$  and the center of  $I_x$  is less than the center of  $I_y$ . The assignment of intervals and tolerances is called a *representation* of the order. A tolerance or bitolerance order is called *bounded* if each tolerance is no more than the corresponding interval length. All known examples of tolerance and bitolerance orders are bounded. Bounded bitolerance orders turn out to be exactly the orders of interval dimension two [3]; the proof of Theorem 1 in this paper suggests why this is so. An *interval graph*, *bounded (min) tolerance (interval) graph*, or *bounded (min) bitolerance (interval) graph* (also known as a *trapezoid graph* [5]) is the incomparability graph of an interval order, a bounded tolerance order, or a bounded bitolerance order. Such a graph or order is called *proper* if it has a representation using intervals no one of which is a proper subset of another, and it is called *unit* if it has a representation using only unit intervals. (Unit interval orders are also known as *semiorders* [16] and unit interval graphs are also known as *indifference graphs* [18].) Tolerance graphs were introduced by Golombic and Monma [11] (as above, we assign intervals and tolerances to vertices and two vertices are adjacent if their intervals overlap by at least one of their tolerances) and studied in more detail by Golombic et al. [12]. This paper asks whether a tolerance graph which has an orientable complement is necessarily bounded; this question is as yet unanswered. Tolerance graphs were put into a more general format by Jacobson et al. [13]. For simplicity of language, we henceforth omit the optional min and interval when we refer to tolerance or bitolerance graphs or orders. It is straightforward to show that a proper tolerance graph (or order) is bounded, so the adjective bounded may be omitted when referring to bounded proper tolerance graphs (or orders) or bounded unit tolerance graphs or (orders).

Fishburn [10, 8, 2] showed that an ordering of  $X$  is an interval ordering if and only if it has no four element restriction isomorphic to the ordering  $\underline{2} + \underline{2}$  illustrated by its covering diagram in Fig. 1, consisting of vertices  $a, b, c$ , and  $d$ , with  $a < b, c < d$ , and no other relations. Scott and Suppes [19] showed that an ordering is a unit interval ordering if and only if it has no four element restriction isomorphic to  $\underline{2} + \underline{2}$  and no four element restriction isomorphic to the ordering  $\underline{3} + \underline{1}$ , also illustrated by its covering diagram in Fig. 1, consisting of four vertices  $a, b, c$ , and  $d$ , with  $a < b < c, a < c$ , and  $d$  incomparable with all of  $a, b$ , and  $c$ . A graph is thus an interval graph if and only if it is a co-comparability graph (the complement of a transitively orientable graph) and has no induced subgraph isomorphic to a four-cycle (the incomparability graph of  $\underline{2} + \underline{2}$ ). The similar characterization of unit interval graphs [18] requires in addition that they have no induced subgraph isomorphic to the ‘claw’  $K_{1,3}$ , the complete bipartite graph with parts of size 1 and 3. There are characterizations of interval graphs and unit interval graphs that do not include the hypothesis that the graph is a cocomparability graph; see e.g., [21].

It is clear that a unit interval order (graph) is proper. Roberts first noted that a proper interval graph (order) is unit. Since a proper interval order contains no

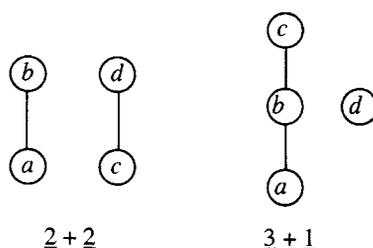


Fig. 1.

restriction isomorphic to  $\underline{3} + \underline{1}$ , this result is an immediate corollary of the Scott–Suppes theorem [19] in the order theoretic context. It is also clear that unit (bi)tolerance orders are proper. Thus it is natural to ask whether proper (bi)tolerance orders are unit. Perhaps surprisingly, the answer is yes for bitolerance orders and no for tolerance orders.

## 2. Proper and unit bitolerance orders

Associated with any bitolerance order on  $X$ , there is a natural linear extension of that ordering: given a representation (in which we may assume, without loss of generality, no two intervals  $I_x$  and  $I_y$  have the same center), we define the linear extension by  $x <_c y$  if and only if the center of  $x$  is less than the center of  $y$ . We refer to this linear ordering as the *central extension* of the representation and denote it with  $<_c$ . Our first two lemmas show that the possible central extensions of a proper bitolerance order are quite restricted.

**Lemma 1.** *Let  $N$  be a four element restriction of a proper bitolerance ordering to four elements  $a, b_1, b_2$ , and  $d$  with  $a > b_1, b_1 < b_2, b_2 > d$  and with no other comparabilities among  $a, b_1, b_2$  and  $d$ . Then in any central extension of the ordering, either  $b_1 <_c d <_c b_2$  or  $b_1 <_c a <_c b_2$  (or both).*

**Proof.** Suppose there is a central extension in which  $d <_c b_1$  and  $a >_c b_2$ . Then, as illustrated in Fig. 2,

$$|I_a \cap I_d| \leq |I_{b_2} \cap I_d| < \min\{t_r(d), t_l(b_2)\} \leq t_r(d)$$

since  $b_2 > d$  and

$$|I_a \cap I_d| \leq |I_{b_1} \cap I_a| < \min\{t_r(b_1), t_l(a)\} \leq t_l(a)$$

since  $a > b_1$ . So  $d <_c a$ , contrary to hypothesis.  $\square$

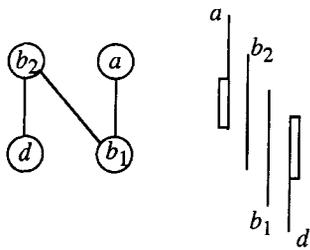


Fig. 2. The rectangles represent the left tolerance of  $a$  and the right tolerance of  $d$ .

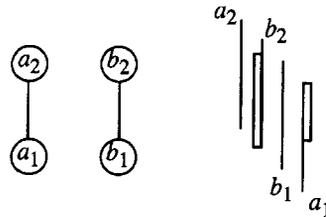


Fig. 3. The rectangles represent the left tolerance of  $b_2$  and the right tolerance of  $a_1$ .

**Lemma 2.** Let  $\underline{2} + \underline{2}$  be a four element restriction of a proper bitolerance ordering to four elements  $a_1, a_2, b_1,$  and  $b_2$  with  $a_1 < a_2, b_1 < b_2$  and no other comparabilities among  $a_1, a_2, b_1,$  and  $b_2$ . Then in any central extension of the ordering, either  $a_1 <_c b_1 <_c b_2 <_c a_2$  or  $b_1 <_c a_1 <_c a_2 <_c b_2$ .

**Proof.** We may assume without loss of generality that  $a_1 <_c b_1$ . Then, as in Fig. 3,

$$|I_{a_1} \cap I_{b_2}| \leq |I_{b_1} \cap I_{b_2}| < \min\{t_r(b_1), t_l(b_2)\} \leq t_l(b_2),$$

since  $b_1 < b_2$ . Then since  $a_1 \not< b_2$ ,

$$|I_{a_1} \cap I_{b_2}| \geq \min\{t_r(a_1), t_l(b_2)\}$$

so that

$$|I_{a_1} \cap I_{b_2}| \geq t_r(a_1).$$

Since  $a_1 < a_2$ ,

$$|I_{a_1} \cap I_{a_2}| < \min\{t_r(a_1), t_l(a_2)\} \leq t_r(a_1);$$

therefore the left endpoint of  $I_{a_2}$  must be greater than that of  $I_{b_2}$ , and since the representation is proper, the centers must be in that order too. Thus  $a_1 <_c b_1 <_c b_2 <_c a_2$ .  $\square$

Lemma 2 appears in graph theoretic terminology (and slightly less generality) as Lemma 5 of [1]. The required linear extensions of the restrictions in Lemmas 1 and 2 characterize proper and unit bitolerance graphs in the following sense.

**Theorem 1.** For an order  $<$  on a set  $X$  the following are equivalent:

1.  $(X, <)$  is a proper bitolerance ordering;
2. The order  $<$  has a linear extension  $<_L$  such that
  - (a) if  $a, b_1, b_2,$  and  $d$  are elements of  $X$  with  $a > b_1, b_1 < b_2, b_2 > d$  and no other comparabilities among these elements, then in  $L$ , either  $b_1 <_L d <_L b_2$  or  $b_1 <_L a <_L b_2$  (or both), and

(b) if  $a_1, a_2, b_1,$  and  $b_2$  are elements of  $X$  with  $a_1 < a_2$  and  $b_1 < b_2$  and no other comparabilities among  $a_1, a_2, b_1,$  and  $b_2$ , then in  $L$ , either  $a_1 <_L b_1 <_L b_2 <_L a_2$  or  $b_1 <_L a_1 <_L a_2 <_L b_2$ .

3.  $(X, <)$  is a unit bitolerance ordering.

**Proof.** Statement 3 implies statement 1 just as for interval orderings. Lemmas 1 and 2 show that statement 1 implies statement 2. Thus we assume that statement 2 holds. We introduce a bit more notation to ease the proof of statement 3. We denote the left endpoint and right endpoint of the interval  $I_x$  by  $le(x)$  and  $re(x)$ , respectively. We define the *left tolerant point* of  $I_x$  by

$$lt(x) = le(x) + t_l(x)$$

and the *right tolerant point* of  $I_x$  by

$$rt(x) = re(x) - t_r(x).$$

Then it is immediate from the definitions that  $x < y$  if and only if,

$$rt(y) < le(x) \quad \text{and} \quad re(x) < lt(y).$$

(Note that this shows that  $x < y$  if and only if all points of the real interval  $[le(x), rt(x)]$  are to the left of all the points of the real interval  $[le(y), rt(y)]$  and all points of  $[lt(y), re(x)]$  are to the left of all points of the real interval  $[lt(y), re(y)]$ ; those familiar with the concept of interval dimension [21] will note this implies bounded bitolerance orders have interval dimension two; the converse is true as well.)

Now suppose the linear ordering of statement 2 is  $x_1 <_L x_2 <_L \dots <_L x_n$ . We choose real numbers

$$le(x_1) < le(x_2) < \dots < le(x_n).$$

Next we choose a real number  $re(x_1)$  greater than  $le(x_n)$ , and we let  $u$  (for unit) equal  $re(x_1) - le(x_1)$ . Now we define the remaining right endpoints by

$$re(x_i) = le(x_i) + u.$$

Finally, we define the left and right tolerant points. First, if  $x_i \not< x_j$  for any  $j$ , define

$$rt(x_i) = \frac{1}{2} [le(x_n) + re(x_1)].$$

Then  $rt(x_i)$  is in each interval  $I_{x_j}$ , so  $x_i \not< x_j$  in this representation for  $j > i$ . (For  $j < i$ , the center of  $x_j$  is less than the center of  $x_i$  so  $x_i \not< x_j$  in this representation.) Otherwise, let  $a$  be the smallest integer such that  $x_i < x_a$  ( $x_a$  is above  $x_i$ ),

$$rt(x_i) = \frac{1}{2} [le(x_a) + le(x_{a-1})]$$

Similarly, if  $x_j \not> x_i$  for any  $i$ , define

$$lt(x_j) = \frac{1}{2} [le(x_n) + re(x_1)].$$

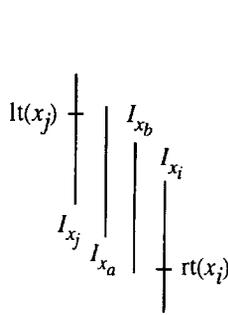


Fig. 4.

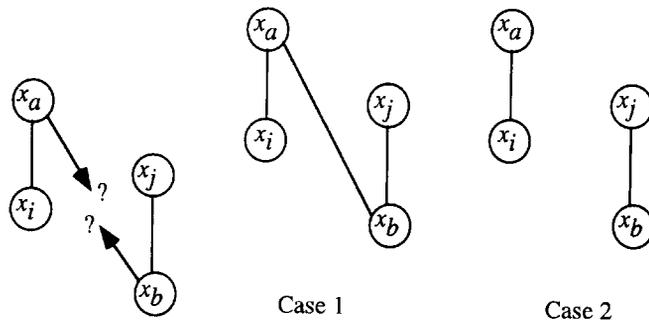


Fig. 5.

Otherwise, let  $b$  be the largest integer such that  $x_b < x_j$  ( $x_b$  is below  $x_j$ ), and let

$$lt(x_j) = \frac{1}{2} [re(x_b) + re(x_{b+1})].$$

By our choices, if  $x_i < x_j$ , then  $rt(x_i) < le(x_a) \leq le(x_j)$  and  $re(x_i) \leq re(x_b) < lt(x_j)$ . So the representation is consistent with  $(X, <)$  in these cases.

If  $x_i$  and  $x_j$  (with  $i < j$ ) are incomparable in  $(X, <)$ , we will assume that the representation gives  $x_i < x_j$  and reach a contradiction. If the representation gives  $x_i < x_j$  then  $rt(x_i) < le(x_j)$  and  $rl(x_i) < lt(x_j)$ . Then, with  $rt(x_i) < le(x_j)$  and  $a$  as defined above,  $le(x_j) > le(x_a)$ . Since  $re(x_i) < lt(x_j)$ , this means that, with  $b$  as defined above,  $re(x_i) < re(x_b)$ . As the illustration in Fig. 4 shows, this means that  $x_a <_L x_j$  and  $x_i <_L x_b$ .

While we know the relationship between  $x_i$  and  $x_j$  in  $(X, <)$ , we do not know the relationships of  $x_a$  and  $x_b$  with  $x_j$  and  $x_i$  respectively and with each other in  $(X, <)$ . However if  $x_b > x_i$ , then by transitivity  $x_j > x_i$ , which is false. Further, if  $x_b < x_i$ , then  $x_b <_L x_i$ , contradicting the inequality  $x_i <_L x_b$  above. Thus  $x_b$  is incomparable to  $x_i$  and similarly,  $x_a$  is incomparable to  $x_j$ . This leaves us two possible orderings among  $x_i, x_j, x_a,$  and  $x_b$ , as illustrated in Fig. 5.

In case 1, part (a) of statement 2 of the theorem tells us that either  $x_b <_L x_j <_L x_a$  or  $x_b <_L x_i <_L x_a$ , contradicting one of  $x_a <_L x_j$  and  $x_i <_L x_b$  above. In case 2, part (b) of statement 2 of the theorem tells us that either  $x_i <_L x_b <_L x_j <_L x_a$  or  $x_b <_L x_i <_L x_a <_L x_j$ , again contradicting  $x_a <_L x_j$  and  $x_i <_L x_b$  from above. Thus  $x_i$  and  $x_j$  must be incomparable in this representation if they are incomparable in  $P$ .  $\square$

### 3. The Fishburn model

Peter Fishburn [9] suggested that the following model of an ordered set would prove interesting for study. We choose, for each  $x$  in a set  $X$ , an interval  $I_x$  and a point  $c_x$  in  $I_x$ . We define  $x < y$  if and only if all the elements of  $I_x$  are less than  $c_y$  and all elements of  $I_y$  are greater than  $c_x$ . Thus our ordering has a tolerance representation in

which the sum of the tolerances  $lt(x)$  and  $rt(x)$  is the interval length  $|I_x|$  for each element  $x$ . We call such a representation a *Fishburn representation*. If an order has a Fishburn representation, then its incomparability graph is a pseudo-interval graph in the sense of [4] and has a natural orientation as an interval catch digraph [17]. Thus an alternate name for a Fishburn representation is an *interval catch representation*.

Fishburn was interested in Fishburn representations with the additional condition that the intervals are all unit intervals; he observed that in this case the orders defined here have the fascinating property that they have no restrictions isomorphic to  $\underline{2} + \underline{3}$  or  $\underline{4} + \underline{1}$ . Our interest is in the more general representation.

Our next theorem appears in [15] (with a geometric proof) as part of Theorem 2.5. Notice that the theorem does not say that a proper or unit bitolerance ordering has a Fishburn representation with unit intervals; the interval bitolerance representation which is unit need not have the sum of the left and right tolerances equal to the interval length.

**Theorem 2.** *An ordering is a proper (unit) bitolerance ordering if and only if it has a Fishburn representation.*

**Proof.** Suppose we have a unit bitolerance representation of an ordering  $(X, <)$  with intervals  $I_x$  of length  $u$  and left and right tolerance  $t_l(x)$  and  $t_r(x)$ . We will change the lengths of the intervals to  $2u - t_l(x) - t_r(x)$ . In particular, if  $I_x = [le(x), re(x)]$ , then we let

$$I'_x = [le(x) + t_l(x) - u/2, re(x) - t_r(x) + u/2] = [le'(x), re'(x)],$$

and we let

$$c_x = \frac{1}{2} [le(x) + re(x)].$$

The reason for these choices is that if  $c_x < c_y$ , then  $I_x$  overlaps  $I_y$  by at least  $t_r(x)$  if and only if, when we increase the right endpoint of  $I_x$  by  $u/2$  and decrease the left endpoint by  $u/2$ , the new (intermediate) intervals, which we denote by  $I_x^*$  and  $I_y^*$ , that we get overlap by  $u + t_r(x)$ . However these intermediate intervals overlap by  $u + t_r(x)$  if and only if when we decrease the right endpoint of  $I_x^*$  by  $t_r(x)$  and increase the left endpoint of  $I_y^*$  by  $t_l(y)$ , the resulting intervals  $I'_x$  and  $I'_y$  overlap by at least  $u - t_l(y)$ . However,  $u - t_l(y)$  is the distance from  $le'(y) = le(y) + t_l(y) - u/2$  to  $\frac{1}{2}[le(y) + re(y)]$ , i.e. the distance from  $le'(y)$  to  $c_y$ . Thus  $I_x$  and  $I_y$  overlap by at least  $t_r(x)$  if and only if  $I'_x$  contains  $c_y$ . Similarly,  $I_x$  and  $I_y$  overlap by at least  $t_l(y)$  if and only if  $I'_y$  contains  $c_x$ . This gives Fishburn representation of  $(X, <)$ . Given a Fishburn representation, we can reverse the construction with any sufficiently large  $u$ .  $\square$

As a corollary we obtain a theorem found in [1].

**Corollary 1.** *An order is a unit tolerance order if and only if it is a 50% tolerance order, i.e. it has a tolerance representation in which the tolerance of each  $x$  is half the length of  $I_x$ .*

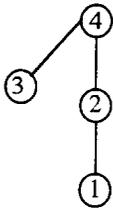
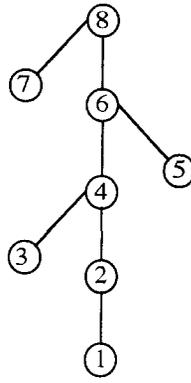
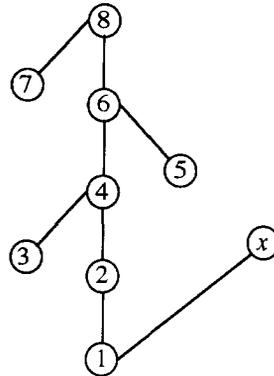


Fig. 6.



(a)



(b)

Fig. 7.

**Proof.** If each  $I_x$  is a unit interval and  $t_1(x) = t_r(x)$ , then  $c_x$  is the center of  $I'_x$  as well as  $I_x$ , and thus  $x < y$  in the Fishburn representation if and only if  $c_x < c_y$  and  $|I'_x \cap I'_y|$  is less than half of either interval length.  $\square$

#### 4. Proper and unit tolerance orders

In contrast to the main theorem of this paper, in [1] there is a family of examples of proper tolerance graphs that are not unit tolerance graphs. In this section we briefly describe the order-theoretic approach to constructing those examples. The basic building block for these orders is the four-element ordered set we refer to as a ‘hook’, shown in Fig. 6 with the vertices numbered in the order of one possible central extension of a tolerance representation.

**Lemma 3.** *In any 50% tolerance representation of the hook in Fig. 6 in which the intervals have centers  $c_1 < c_2 < c_3 < c_4$ ,  $c_4 - c_3 > c_3 - c_1$ .*

**Proof.** Since vertices 3 and 1 are incomparable, either  $c_1 \in I_3$  or  $c_3 \in I_1$ . But  $c_3 > c_2$  and  $c_2$  is greater than all members of  $I_1$ , so  $c_1 \in I_3$ . Thus  $c_3 - c_1$  is less than the half-length  $c_3 - \text{le}(3)$  of  $I_3$ . Since  $c_4$  is greater than all members of  $I_3$ ,  $c_4 - c_3$  is greater than this half-length, and the conclusion follows.  $\square$

Now suppose that, as in Fig. 7, we have additional vertices 5–8 such that 3–6 form a hook and 5–8 form a hook. Note that we do not require that our ordered set be isomorphic to that of Fig. 7, for example vertex 5 could be over vertex 1 without violating the restrictions that the three sets of vertices given are hooks.

Then

$$\begin{aligned}
 c_8 - c_5 &= c_8 - c_7 + c_7 - c_5 \\
 &> c_7 - c_5 + c_6 - c_5 \\
 &> c_6 - c_5 + c_6 - c_5 \\
 &> c_5 - c_3 + c_5 - c_3 \\
 &> c_5 - c_3 + c_4 - c_3 \\
 &> c_5 - c_3 + c_3 - c_1 = c_5 - c_1
 \end{aligned}$$

This gives us

**Lemma 4.** *Suppose that  $c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8$  is a central extension of a 50% tolerance representation of an order on  $\{1, 2, 3, 4, 5, 6, 7, 8\}$  in which  $\{1, 2, 3, 4\}$ ,  $\{3, 4, 5, 6\}$  and  $\{5, 6, 7, 8\}$  form hooks. Then  $c_8 - c_5 > c_5 - c_1$ .*

Now suppose we add one more vertex  $x$  above vertex 1 as in Fig. 7(b) and seek a 50% tolerance representation in which  $c_x < c_5$ . Then the half length of  $I_x$  is less than  $c_5 - c_1$ , and therefore the right endpoint of  $x$  is less than  $c_8$ , so that  $x \prec 8$ . This proves the following lemma

**Lemma 5.** *An order on  $\{1, 2, 3, 4, 5, 6, 7, 8, x\}$  in which*

- $\{1, 2, 3, 4\}$ ,  $\{3, 4, 5, 6\}$  and  $\{5, 6, 7, 8\}$  are hooks,
- $x \succ 1$ , and
- $x \not\prec 8$

*has no 50% tolerance representation in which  $c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8$  and  $c_x < c_5$ .*

Of course it is not yet clear whether there is a proper representation of an ordered set with a proper tolerance representation as described in Lemma 5. However it is possible to reduce this question to a straightforward matter of linear programming. Note that vertex 2 cannot be greater than vertex 3 because of the order of their centers. If vertex 3 is incomparable with vertex 1, then it cannot be over vertex 2 either. Thus, if we can make vertices 1 and 3 incomparable, make vertex 1 less than vertex 2, and make vertices 2 and 3 less than vertex 4, we have all the comparabilities of the hook. The statement that vertex  $i$  is over vertex  $j$  can be described by inequalities by

$$le(i) > re(j) - t(j) \quad \text{and} \quad re(j) < le(i) + t(i).$$

By scaling we may assume that each strict inequality must be strict by at least one unit. So these become

$$le(i) \geq re(j) - t(j) + 1 \quad \text{and} \quad re(j) \leq le(i) + t(i) - 1.$$

The fact that the representation is proper and the statement that  $c(i) > c(j)$  may be expressed in inequalities by

$$re(i) \geq re(j) + 1, \quad le(i) \geq le(j) + 1.$$

When the center of  $i$  is above the center of  $j$ , the statement that  $i$  and  $j$  are incomparable may be expressed as one of the two inequalities

$$re(j) - le(i) \geq t(i) \quad \text{or} \quad re(j) - le(i) \geq t(j)$$

Note that in the hook on  $\{1, 2, 3, 4\}$ , the incomparability of vertices 1 and 3 may only be expressed by the inequality

$$re(1) - le(3) \geq t(3),$$

because if the overlap of vertices 1 and 3 were greater than  $t(1)$ , so would be the overlap of vertices 1 and 2, and this is impossible since  $1 < 2$  in the hook. This means that the only inequality we may use to express the incomparability of the first and third elements in a hook is  $re(i) - le(j) \geq t(j)$  (with  $j = i + 2$ ). Thus all the required relationships among the vertices in  $\{1, 2, 3, 4, 5, 6, 7, 8\}$  may be expressed as a system of linear inequalities. The only other relationships specified in Lemma 5 were that  $x > 1$ ,  $c_x < c_5$ , and  $x$  is incomparable to vertex 8 (since  $c_x < c_5$ ,  $x \not< 8$  is equivalent to  $x$  and 8 being incomparable). The first two relationships may be expressed as inequalities as above. Since  $c_x < c_5$ , the overlap of  $I_8$  and  $I_x$  must be less than that of  $I_8$  and  $I_5$ , so the overlap of  $I_8$  and  $I_x$  cannot be more than  $t_8$  because then vertex 5 would not be less than vertex 8. Thus there is only one inequality that expresses the fact that vertices  $x$  and 8 are incomparable. Therefore there is no system of linear equalities which captures all the required relationships among the vertices  $\{1, 2, \dots, 8, x\}$ . Using Lindo, we minimized the difference  $re(8) - le(1)$  subject to the inequalities that describe the conditions in Lemma 5, and we obtained the following intervals and tolerances:

$$\begin{aligned} I_1 &= [0, 11], & t_1 &= 11, & I_2 &= [1, 12], & t_2 &= 11, & I_3 &= [2, 19], & t_3 &= 9, \\ I_4 &= [11, 20], & t_4 &= 9, & I_5 &= [13, 22], & t_5 &= 6, & I_6 &= [17, 23], & t_6 &= 6, \\ I_7 &= [18, 24], & t_7 &= 4, & I_8 &= [21, 25], & t_8 &= 4, & I_x &= [12, 21], & t_x &= 0. \end{aligned}$$

This gives us the ordered set shown in Fig. 8(a). This does not complete the proof that our example is proper but not unit, because there could be (and is) a unit representation of this order in which the centers of the intervals do not lie in the specified order. Thus we ask, in light of Lemmas 1 and 2 if it is possible to add vertices to Fig. 8 in such a way that in any proper tolerance representation, the centers for the intervals must occur in the order we specified. The answer is that adding the two vertices  $y$  and  $z$  shown in Fig. 8(b) is sufficient. To prove this we need one more lemma, analogous to Lemmas 1 and 2, but valid only for tolerance orders.

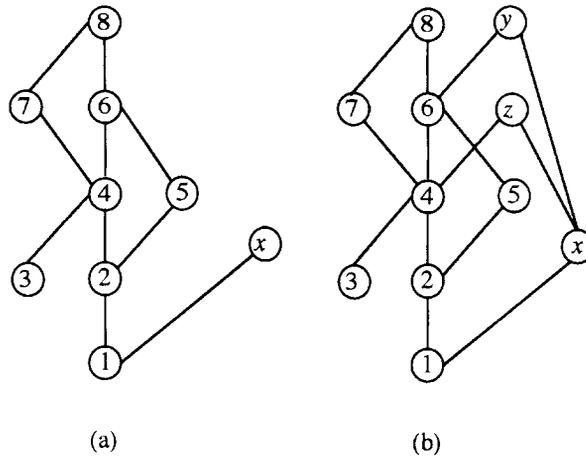


Fig. 8.

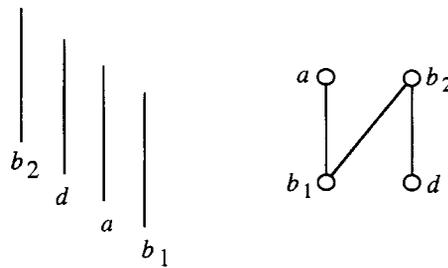


Fig. 9.

**Lemma 6.** *In any proper tolerance representation of the four element set  $\{a, b_1, b_2, d\}$  with  $a > b_1, b_1 < b_2, b_2 > d$ , and no other comparabilities,  $c_d < c_a$ .*

**Proof.** If we assume the contrary, then as illustrated in Fig. 9, we have that  $c_{b_1} < c_a < c_d < c_{b_2}$ . Now since  $a > b_1$  we have  $|I_a \cap I_{b_1}| < t_a$ . Now  $a$  and  $b_2$  are incomparable, but the overlap of  $I_a$  and  $I_{b_2}$  is less than that of  $I_d$  and  $I_{b_2}$ , so  $|I_a \cap I_{b_2}|$  cannot be greater than  $t_{b_2}$ . Thus  $t_a < |I_a \cap I_{b_2}|$ . Using once again that the overlap of  $I_a$  and  $I_{b_2}$  is less than that of  $I_d$  and  $I_{b_2}$ , gives us

$$|I_a \cap I_{b_1}| < t_a < |I_{b_2} \cap I_a| < |I_d \cap I_{b_2}|.$$

By symmetry we have

$$|I_d \cap I_{b_2}| < t_d < |I_{b_1} \cap I_d| < |I_a \cap I_{b_1}|,$$

and by transitivity,  $|I_a \cap I_{b_1}| < |I_a \cap I_{b_1}|$ , a contradiction. Thus  $c_d$  must be less than  $c_a$  after all.  $\square$

**Theorem 3.** *There is no unit tolerance representation of the order shown in Fig. 8(b).*

**Proof.** We show that in any central extension of a proper representation of the order,  $c_i < c_{i+1}$  for  $i$  between 1 and 7, and  $c_x < c_5$ . First we apply the Lemma 6 to the ordered set  $\{x, z, 2, 5\}$  to obtain  $c_x < c_5$ . Now we apply Lemma 2 to the ordered set  $\{x, y, 7, 8\}$  to obtain that  $c_y > c_8$ . Next we apply Lemma 1 three times in succession to the ordered sets  $\{6, 7, 8, y\}$ ,  $\{4, 5, 6, 7\}$ , and  $\{2, 3, 4, 5\}$  to obtain that  $c_6 < c_7 < c_8$ , then  $c_4 < c_5 < c_6$ , and finally  $c_2 < c_3 < c_4$ . This completes the proof.  $\square$

Finally there is the question of whether adding  $y$  and  $z$  to the ordered set in Fig. 8(a) results in an ordered set with a proper tolerance representation.

**Theorem 4.** *There is a proper tolerance representation of the order shown in Fig. 8(b).*

**Proof.** To the interval representation obtained for the order of Fig. 8(a), add the interval  $[24, 25]$  for  $y$  and  $[22, 25]$  for  $z$  and give each of  $y$  and  $z$  tolerance 0.  $\square$

We would like to thank Lecretia Wilson for setting up and solving the linear programming problem described above.

## 5. Graph theoretic interpretation

There is a natural interpretation of Theorem 1 in terms of ordered subgraphs of ordered graphs. We adapt the definitions of ordered graphs in [6, 7, 14] as follows: An *ordered graph*  $G = (V, E, L)$  is a graph  $(V, E)$  and a linear ordering  $L$  of the vertices of  $G$ . An *ordered induced subgraph* of  $G$  is an (induced) subgraph of  $(V, E)$  together with its inherited ordering. Two ordered graphs are isomorphic if there is a bijection  $\varphi$  between the vertex sets such that  $\varphi$  and  $\varphi^{-1}$  are order preserving functions that preserve adjacency as well.

As an example (already observed in [6, 7, 14]) of how this concept may be used, saying that  $G = (V, E)$  is the incomparability graph of an ordered set with linear extension  $L$  is the same as saying that the ordered graph  $G' = (V, E, L)$  has no induced 3-element ordered subgraph isomorphic to the ordered graph shown in Fig. 10. (Note that for the graph in Fig. 10 to be the complement of the comparability graph of an order with linear extension  $1 < 2 < 3$ , the order would have to have  $1 < 2$ ,  $2 < 3$ , and  $1 \not< 3$ , an impossibility.)

How do we express the conditions of Theorem 1 in this language? Note, for example, that the conclusion of condition 2(a) of Theorem 1 can be rewritten to say that  $L$  is not the ordering  $a <_L b_1 <_L b_2 <_L d$ . Thus the condition forbids a certain linear extension of the vertex set of the incomparability graph of the ordered set in Fig. 2. This leads us to the following theorem in which we use the notation  $L: a, b, c, d$

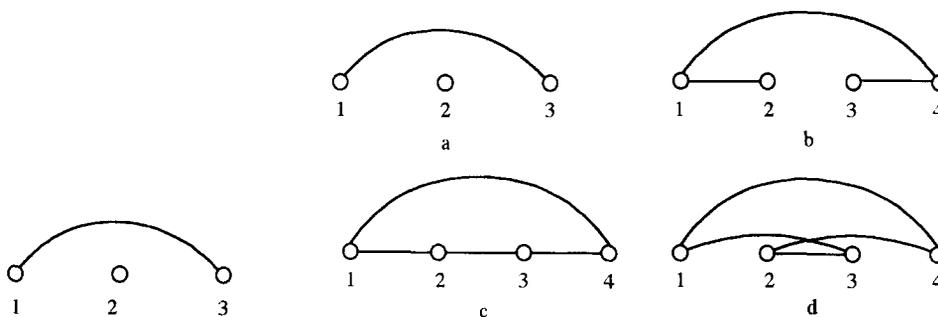


Fig. 10.

Fig. 11. The ordered graphs of Theorem 5.

to mean that  $L$  is a linearly ordered set in which  $a$  precedes  $b$ , which precedes  $c$ , which precedes  $d$ . (We suggest reading the colon as ‘is given by’.)

**Theorem 5.** *The following statements about a graph  $G = (X, E)$  are equivalent.*

1.  $G$  is a proper bitolerance graph
2. There is a linear ordering  $L: x_1, x_2, x_3, x_4$  of the vertices of  $G$  such that the ordered graph  $G = (X, E, L)$  has no ordered subgraph isomorphic to any of the four ordered graphs  $H = (V, F, M)$  described below and shown in Fig. 11.
  - (a)  $V = \{1, 2, 3\}$ ,  $F = \{(1, 3)\}$ ,  $M: 1, 2, 3$ ;
  - (b)  $V = \{1, 2, 3, 4\}$ ,  $F = \{(1, 2), (3, 4), (1, 4)\}$ ,  $M: 1, 2, 3, 4$ ;
  - (c)  $V = \{1, 2, 3, 4\}$ ,  $F = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$ ,  $M: 1, 2, 3, 4$ ;
  - (d)  $V = \{1, 2, 3, 4\}$ ,  $F = \{(1, 3), (2, 3), (2, 4), (1, 4)\}$ ,  $M: 1, 2, 3, 4$ .
3.  $G$  is a unit bitolerance graph.

**Proof.** We use the notation ‘condition  $a.b$ ’ to stand for condition  $b$  of Theorem  $a$ , and the notation ‘condition  $a.b.c$ ’ to stand for condition  $c$  of part  $b$  of Theorem  $a$ . We already know that  $G$  satisfies condition 5.1 (5.3) if and only if there is an orientation of the complement  $\overline{G}$  that satisfies condition 1.1 (1.3). Thus it suffices to show that a graph  $G$  satisfies condition 5.2 if and only if its complement has an orientation which is an ordering satisfying condition 1.2.

We noted above that there is a linear ordering of the vertices of  $G$  satisfying condition 5.2.a if and only if  $\overline{G}$  has an orientation which is an ordering of  $X$  with the given linear ordering as a linear extension. We now show that  $G$  and a linear ordering of its vertices satisfying condition 5.2.a also satisfy condition 5.2.b if and only if a corresponding orientation of  $\overline{G}$  as an ordering satisfies condition 1.2.a, and that  $G$  and a linear ordering of its vertices satisfying condition 5.2.a also satisfy condition 5.2.c and 5.2.d if and only if a corresponding orientation of  $\overline{G}$  satisfies condition 1.2.b.

Suppose now that  $G$  and a linear ordering of its vertices satisfy condition 5.2.a and that the ordering  $P$  is an orientation of  $\overline{G}$ . (Equivalently, we could assume

that  $P$  is an ordering of  $X$  with linear extension  $L$ , and that  $G$  is its incomparability graph.)

Now suppose further that  $P$  satisfies condition 1.2.a. As noted above, this means that the restriction of the ordering  $L$  and  $V = \{a, b_1, b_2, d\}$  is not the ordering  $M: a, b_1, b_2, d$ . But the edge set of the induced subgraph of  $G$  on  $V$  is  $F = \{(a, b_2), (b, d), (a, d)\}$ . Thus condition 1.2.a implies that the ordered graph  $G = (X, E, L)$  does not have the ordered subgraph  $H = (V, F, M)$ . Further, any ordered subgraph of  $G$  isomorphic to  $F$  will have only orientations isomorphic to the restriction of  $P$  to  $V$  or its dual (all order relations reversed). Since  $H$  is isomorphic to the graph of condition 5.2.b, this proves that  $G$  satisfies condition 5.2.b. The converse may be proved by reversing the argument.

Now suppose instead that  $P$  satisfies condition 1.2.b. This implies that the restriction of  $L$  to the set  $V = \{a_1, a_2, b_1, b_2\}$  is not  $M: a_1, b_1, a_2, b_2$ ,  $M: b_1, a_1, b_2, a_2$ ,  $M: a_1, a_2, b_1, b_2$ , or  $M: b_1, b_2, a_1, a_2$ . But the edge set of the induced ordered subgraph of  $G$  on the set  $V = \{a_1, a_2, b_1, b_2\}$  is  $F = \{(a_1, b_1), (b_1, a_2), (a_2, b_2), (a_1, b_2)\}$ . The ordered graphs with  $M: a_1, b_1, a_2, b_2$  and  $M: b_1, a_1, b_2, a_2$  are isomorphic to the ordered graph of condition 5.2.c, and the ordered graphs with  $M: a_1, a_2, b_1, b_2$  and  $M: b_1, b_2, a_1, a_2$  are isomorphic to the ordered graph of condition 5.2.d. Further, any orientation of the complement of  $H = (V, F, M)$  is  $P$  or an order obtained by reversing one or both of the order relations of  $P$  (which yields an ordering isomorphic to  $P$ ). Again, the four forbidden linear extensions will yield ordered graphs isomorphic to those of conditions 5.2.c or 5.2.d. Thus  $G$  satisfies conditions 5.2.c and 5.2.d. The converse may be proved by reversing the argument.  $\square$

This theorem does not simplify the recognition problem for proper or unit bitolerance graphs, because as shown in [7], given an ordered graph  $H$  the decision problem of whether there is an ordering of the vertex set of a graph  $G = (V, E)$  so that the resulting ordered graph does not contain  $H$  as an ordered subgraph is NP-complete for a wide variety of graphs  $H$ , including that of part 2(d) of Theorem 5. However this does not mean that the recognition problem for proper bitolerance graphs is NP-complete, because it involves recognizing graphs that exclude all the ordered graphs of Theorem 5 as ordered subgraphs. This leaves us with a tantalizing problem; what is the complexity status of recognizing proper bitolerance graphs?

The recognition problem is especially interesting in light of the recent result of Shull and Trenk [20] that the directed graph concepts of proper bitolerance digraphs, unit bitolerance digraphs, and interval catch digraphs are all equivalent. Interval catch digraphs are recognizable in polynomial time [17], so this solves the directed version of the recognition problem. While a representation of a bitolerance graph yields a bitolerance digraph in a natural way, without a representation, we have as yet no way to associate with a possible bitolerance graph a digraph to test to see if it is an interval catch digraph. Thus we cannot take advantage of Shull and Trenk's result to aid in the recognition of proper bitolerance graphs.

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