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A CLASSIFICATION OF CERTAIN GRAPHS WITH MINIMAL IMPERFECTION PROPERTIES

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The family of (α, ω) graphs are of interest for several reasons. For example, any minimal counter-example to Berge's Strong Perfect Graph Conjecture belongs to this family. This paper accounts for all $(4, 3)$ graphs. One of these is not obtainable by existing techniques for generating $(\alpha+1, \omega)$ graphs from (α, ω) graphs.

1. Introduction

A graph G is said to be *perfect* if for each induced subgraph G' of G the size of the largest clique of G' is equal to the chromatic number of G'. The Strong Perfect Graph Conjecture of Berge asserts that a graph is perfect if and only if it contains no induced subgraphs which are holes or antiholes, where a *hole* is a chordless cycle of odd length at least 5, and an *antihole* is the complement of a hole. Lovász [12, 13] proved a weaker conjecture of Berge: a graph is perfect if and only if its complement is perfect.

The Strong Perfect Graph Conjecture has been established for several classes of graphs including: planar graphs [16], circular arc graphs [17], $K_{1,3}$-free graphs [15, 19], 3-chromatic graphs [18], and graphs with maximum degree at most 6 [7].

Another way to state the Strong Conjecture is to say that any imperfect graph whose proper induced subgraphs are all perfect must be either a hole or an antihole. Padberg [14] showed that such a minimally imperfect, or *critical*, graph must be an (α, ω) *graph*, defined below.

Definition. G is an (α, ω) graph if, and only if,

(i) its vertex set has size $\alpha\omega + 1$;

(ii) its largest stable set has size α, and its largest clique has size ω;

(iii) each vertex is in precisely α stable sets of size α and ω cliques of size ω;

(iv) each clique of size ω is disjoint from precisely one stable set of size α, and each stable set of size α is disjoint from precisely one clique of size ω.

In what follows, n denotes $\alpha\omega + 1$.

An (α, ω) graph is said to be *normalized* if each of its edges belongs to at least

one clique of size ω. Each (α, ω) graph contains a unique normalized (α, ω) subgraph, because removing edges which belong to no cliques of size ω does not create any new stable sets of size α [6, 18].

The paper of Chvátal, Graham, Perold, and Whitesides [6] establishes two additional contexts in which (α, ω) graphs arise. First of all, there is a correspondence between normalized (α, ω) graphs and solutions to the system of equations

$$JX = XJ = \alpha J, \qquad JY = YJ = \omega J, \qquad XY = J - I, \tag{1}$$

where X and Y are matrices of 0's and 1's, J has all entries 1, I is the identity matrix, and all these matrices are $n \times n$. Bridges and Ryser [4] call the above matrices X and Y an $(n, 0, 1)$ system on α, ω. Second, there is a correspondence between normalized (α, ω) graphs and packings of the complete graph K_n by complete bipartite graphs $K_{\alpha,\omega}$ with each edge of K_n covered exactly twice. C. Huang [9, 10] and C. Huang and Rosa [8] have studied such packings.

The graphs denoted $C_n^{\omega-1}$ are (α, ω) graphs; they have vertices $v_0, \ldots, v_{\alpha\omega}$, with v_i adjacent to v_j whenever there is a d such that $0 < d < \omega$ and $d \equiv i - j$ or $j - i \pmod n$. Holes and antiholes are of this type. In [6], methods are given for constructing (α, ω) graphs which are not of this type.

The purpose of this paper is to describe all normalized (4, 3) graphs. One of these is a graph which is neither C_{13}^2 nor a graph obtainable by the methods of [6]. Of course, none of these graphs is a counterexample to the Strong Perfect Graph Conjecture, as Tucker [18] has shown the conjecture holds for graphs with $\omega \leq 3$.

2. Properties of (α, ω) graphs

We now list several well known properties of (α, ω) graphs which we will use frequently throughout this paper. For convenience, we will use the work *clique* (*stable set*) to refer to a clique (stable set) of maximum size only.

Lemma 1. *If G is an (α, ω) graph, then G contains exactly $\alpha\omega + 1$ cliques (of size ω) and exactly $\alpha\omega + 1$ stable sets (of size α).*

Proof. This follows easily from the definition of an (α, ω) graph.

Lemma 2. *Let G be an (α, ω) graph. Let its cliques be $T_1, \ldots, T_{\alpha\omega+1}$ and its stable sets be $S_1, \ldots, S_{\alpha\omega+1}$, where $T_i \cap S_j = \emptyset$ if and only if $i = j$. If vertex v belongs to $T_{a_1}, \ldots, T_{a_\omega}$, then the stable sets $S_{a_1}, \ldots, S_{a_\omega}$ partition $G - v$. Similarly, if v belongs to $S_{b_1}, \ldots, S_{b_\alpha}$, then the cliques $T_{b_1}, \ldots, T_{b_\alpha}$ partition $G - v$.*

Proof. Let X be the matrix whose rows are the incidence vectors of the stable sets $S_1, S_2, \ldots, S_{\alpha\omega+1}$, and let Y be the matrix whose columns are the incidence

vectors of the cliques $T_1, T_2, \ldots, T_{\alpha\omega+1}$. Then equation (1) holds, and also

$$YX = X^{-1}XYX = X^{-1}(J-I)X = X^{-1}JX - I = J - I. \tag{2}$$

The Lemma now follows from the fact that $YX = J - I$.

Remark 1. A consequence of Lemma 2 is that the *pseudo p-critical* graphs of Tucker [18] are precisely the (α, ω) graphs.

Lemma 3. *Let G be a graph, and define a graph $M(G)$ by making the vertices of $M(G)$ correspond to the cliques of G and making vertices in $M(G)$ adjacent whenever the corresponding cliques intersect. If G is an (α, ω) graph, then so is $M(G)$.*

Proof. See [18].

3. Generation of (4, 3) graphs from (3, 3) graphs

We assume throughout the rest of this paper that G is a normalized (4, 3) graph whose cliques are "triangles" $T_1, \ldots, T_{13}$ and whose stable sets are $S_1, \ldots, S_{13}$, where $S_i \cap T_j = \emptyset$ if and only if $i = j$. By an $i_1, \ldots, i_k$*-vertex*, we mean a vertex which belongs to the stable sets $S_{i_1}, \ldots, S_{i_k}$. By $\triangle$xyz, we mean a triangle whose vertices are x, y, and z.

Remark 2. We emphasize that Lemma 2 says the following: if a vertex v belongs to distinct T_{i_1}, T_{i_2}, and T_{i_3}, then each other vertex of G is exclusively an i_1-vertex, an i_2-vertex, or an i_3-vertex. Also, it says that if distinct T_j and T_k intersect, then there are no j, k-vertices.

We define a graph $K(G)$ from G as follows. We make the vertices of $K(G)$, like the vertices of $M(G)$, correspond to the triangles of G. This time, however, we make vertices adjacent whenever the triangles to which they correspond intersect in an edge.

We first show that the maximum degree of $K(G)$ is at most 2. Then we use this fact to prove that $K(G)$ contains a path of length at least 5 if and only if G can be generated from a (3, 3) graph by the first construction method of Chvátal [6].

Lemma 4. *The graph $K(G)$ has maximum degree at most* 2.

Proof. If a triangle of G met three other triangles in edges, then the four triangles together would give rise to a clique of size 4 in $M(G)$. However, $M(G)$ is a (4, 3) graph according to Lemma 3.

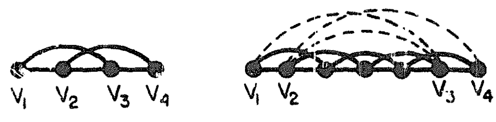

Fig. 1. Configuration replacement generating a (4, 3) graph. An edge shown dashed is to be included provided that in H, it belongs to a triangle whose third vertex is in $H-\{v_1, v_2, v_3, v_4\}$.

Lemma 5. *No edge of G is in three triangles.*

Proof. Suppose triangle T_i with vertices a, b, and c shared an edge ab with triangles T_j and T_k. Then since T_i would intersect S_j and S_k, c would be a j, k-vertex, which is ruled out by Lemma 2.

Lemma 6. Suppose that H is a $(3, 3)$ graph containing the configuration of four vertices shown in Fig. 1(a) and that either $\{v_1, v_2, v_3\}$ is one of three cliques partitioning $H-v_4$ or $\{v_2, v_3, v_4\}$ is one of three cliques partitioning $H-v_1$. Then replacing the configuration shown in Fig. 1(a) by the configuration shown in Fig. 1(b) generates a $(4, 3)$ graph.

Proof. This is the construction of [6] specialized to the generation of $(4, 3)$ graphs from $(3, 3)$ graphs.

Lemma 7. *If $K(G)$ contains a path of length at least 5, then G contains the configuration of triangles shown in Fig. 2, and*

(i) d *is the only common neighbor of* b *and* f;

(ii) *vertices* b *and* g *are adjacent (similarly for* a *and* f).

Proof. If $K(G)$ contains a path of length at least 5, then Lemma 5 implies that G contains the configuration in Fig. 2.

(i) Suppose that T_6 is the third triangle containing b and that T_7 is the third triangle containing f. Then each vertex v distinct from b and f must be an i, j-vertex for some i in $\{1, 2, 6\}$ and j in $\{4, 5, 7\}$. In particular, if v is a 3-vertex, then by Remark 2 it must be a 3, 6, 7-vertex, as T_3 intersects T_1, T_2, T_4, and T_5. Hence T_6 and T_7 do not intersect, again by Remark 2. Therefore, b and f have no common neighbors other than d.

We leave to the reader the proof of part (ii), which can be handled by repeated applications of Remark 2.

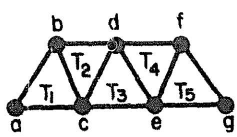

Fig. 2. Configuration contained in G, where $K(G)$ contains a path of length 5. Some adjacencies may not be shown.

Lemma 8. *If $K(G)$ contains a path of length at least 5, then G can be obtained by the construction in Lemma 6.*

Proof. Assume G contains the configuration shown in Fig. 2. Let H be the graph obtained from G by removing c, d, and e, making b adjacent to f and g, and making a adjacent to f. Triangles $T_1, \ldots, T_5$ are destroyed by the removal of c, d, and e. If ag were an edge of G, then by Lemma 7, af and bg would also be edges. Then a, …, g would form a connected component of G, and the remaining five vertices could contain at most three triangles. Hence ag is not an edge of G, and vertices a, b, f, and g do not form a clique in H. It is easy to check, using Lemma 7, that $\triangle$abf and $\triangle$bfg are the only triangles created. Consequently H has clique size 3 and contains ten triangles. Also, each vertex of H belongs to three triangles.

Consider how the stable sets of G intersect {a, …, g}. With the exception of S_2, any stable set containing a contains d. Hence three stable sets contain both a and d. Similarly, three stable sets contain both d and g. Since vertex d belongs to S_1 and S_5, it follows that the two remaining stable sets to which d belongs both contain a and g. Further consideration along these lines shows that G has, in addition to $S_1, \ldots, S_5$ and the two stable sets containing a, d, and g, three stable sets containing b and e and three stable sets containing c and f. This accounts for all thirteen of the stable sets S_i of G. This information, along with other information we are about to produce, is represented in Fig. 3.

With the exception of S_3, each S_i loses a vertex of T_3 in the formation of H from G. The description of the stable sets of G shows that the remaining vertices of S_i, $i \neq 3$, continue to be mutually non-adjacent, as no stable set of G contains both b and g or both a and f. We will call the set of remaining vertices of S_i, $i \neq 3$, the *image* of S_i. Since S_3 is the only stable set of G containing both b and f, the

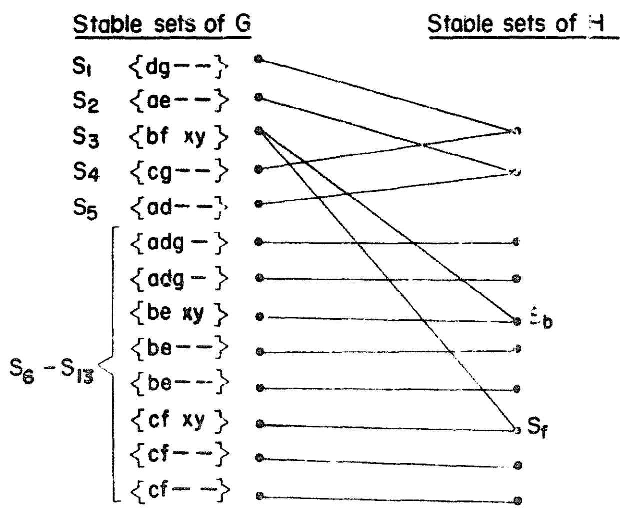

Fig. 3. Stable sets of G and H, where $K(G)$ has a path of length 4.

joining of b to f produces in H two stable sets S_b and S_f of size 3 containing b and f, respectively. For future reference, let $S_3=\{b, f, x, y\}$, and note that $S_b=\{b, x, y\}$ and $S_f=\{f, x, y\}$. Also, note that $\{b, e, x, y\}$ is a stable set (one of the $S_6, \ldots, S_{13}$) whose image is S_b and that $\{c, f, x, y\}$ is a stable set (one of the $S_6, \ldots, S_{13}$) whose image is S_f.

We now check that the stable sets of H have the properties of stable sets in (3, 3) graphs. Clearly, the stability number of H is 3. The vertices of each stable set of H together with one of c, d, e form a stable set of G. By Remark 2, each vertex of G except c is an i-vertex for exactly one i in $\{1, 2, 3\}$. Each vertex except d is a j-vertex for exactly one j in $\{2, 3, 4\}$. Consequently, each 1-vertex except d is also a 4-vertex. It follows that S_1 and S_4 have the same image. Similarly, S_2 and S_5 have the same image. As previously declared, $\{b, e, x, y\}$ has image S_b and $\{c, f, x, y\}$ has image S_f. By referring to the list of stable sets of G given in Fig. 3, we can now deduce that there are no additional agreements among the images of these stable sets. This fact is depicted in Fig. 3. Hence H contains exactly ten stable sets. Each vertex of H is a 1, 4-vertex, a 2, 5-vertex, or a 3-vertex of G. By considering the vertices type by type, we can check that each vertex of H lies on at most three stable sets of H. Then counting vertex-stable set incidences in H shows that each vertex lies in exactly three stable sets.

Next, we check that each of the ten triangles of H is disjoint from exactly one stable set of H, and vice versa. The common image of S_1 and S_4 is the only stable set of H disjoint from $\triangle$abf, and the common image of S_2 and S_5 is the only stable set disjoint from $\triangle$bfg. For $6 \leq i \leq 13$, T_i belongs to H, and T_i is disjoint from a stable set of H if and only if that stable set is the image of the stable set S_i disjoint from T_i in G. Thus, each triangle of H is disjoint from a unique stable set. Since the images of $S_6, \ldots, S_{13}$ together with the images of S_1 and S_2 account for all the stable sets of H, each stable set of H is disjoint from at least one triangle of H. It follows that the property of being disjoint pairs the triangles and stable sets of H.

We have now shown that H is a (3, 3)-graph. By Lemma 2, H–a is partitioned by the triangles of H disjoint from the stable sets of H containing a. Since the common image of S_2 and S_5 contains a and is disjoint from $\triangle$bfg, $\triangle$bfg is in this

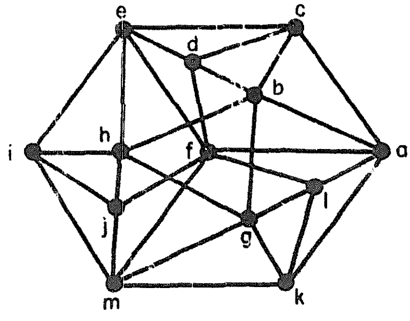

Fig. 4. The normalized (4, 3) graph not constructible by the method of [6].

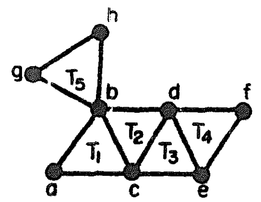

Fig. 5. Configuration contained in G, where $K(G)$ has maximum path length 4. Some adjacencies may not be shown.

partition of H−a by cliques. It follows that we can generate G from H by the construction of Lemma 6 by choosing v_1, v_2, v_3 and v_4 to be a, b, f, and g, respectively.

4. The (4, 3) graphs

The (4, 3) graphs which are obtainable from (3, 3) graphs by the method of Lemma 6 are listed in [6]. One of these graphs was discovered independently by H.-C. Huang [3], [11]. The following theorem completes the description of *all* normalized (4, 3) graphs.

Theorem. *The normalized* (4, 3) *graphs consist of the graph shown in Fig.* 4 *together with those graphs which can be constructed from* (3, 3) *graphs by the method of Lemma* 6.

Proof. Suppose G is a normalized (4, 3) graph which cannot be constructed from a (3, 3) graph by the method of Lemma 6. By repeated applications of Remark 2, one can check case by case that $K(G)$ cannot have maximum path length equal to 1, 2, or 3. Then by Lemma 8, $K(G)$ has maximum path length 4. We may assume that the configuration shown in Fig. 5 appears in G, where a, . . . , h are distinct vertices. It is straightforward to check that this configuration gives rise to only one normalized (4, 3) graph G with $K(G)$ having maximum path length 4.

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