

Dartmouth College

## Dartmouth Digital Commons

---

Open Dartmouth: Published works by  
Dartmouth faculty

Faculty Work

---

6-3-2005

### Monomial Nonnegativity and the Bruhat Order

Brian Drake  
*Brandeis University*

Sean Gerrish  
*University of Michigan-Ann Arbor*

Mark Skandera  
*Dartmouth College*

Follow this and additional works at: <https://digitalcommons.dartmouth.edu/facoa>



Part of the [Discrete Mathematics and Combinatorics Commons](#)

---

#### Dartmouth Digital Commons Citation

Drake, Brian; Gerrish, Sean; and Skandera, Mark, "Monomial Nonnegativity and the Bruhat Order" (2005).  
*Open Dartmouth: Published works by Dartmouth faculty*. 2326.  
<https://digitalcommons.dartmouth.edu/facoa/2326>

This Article is brought to you for free and open access by the Faculty Work at Dartmouth Digital Commons. It has been accepted for inclusion in Open Dartmouth: Published works by Dartmouth faculty by an authorized administrator of Dartmouth Digital Commons. For more information, please contact [dartmouthdigitalcommons@groups.dartmouth.edu](mailto:dartmouthdigitalcommons@groups.dartmouth.edu).

# Monomial nonnegativity and the Bruhat order

Brian Drake, Sean Gerrish, Mark Skandera

Dept. of Mathematics, Brandeis University  
MS 050, P.O. Box 9110, Waltham, MA 02454  
bdrake@math.brandeis.edu

Dept. of Mathematics, University of Michigan  
2074 East Hall, Ann Arbor, MI 48109-1109  
sgerrish@umich.edu

Dept. of Mathematics, Dartmouth College  
6188 Bradley Hall, Hanover, NH 03755-3551  
mark.skandera@dartmouth.edu

Submitted: Mar 11, 2005; Accepted: May 6, 2005; Published: Jun 3, 2005

MR Subject Classifications: 15A15, 05E05

## Abstract

We show that five nonnegativity properties of polynomials coincide when restricted to polynomials of the form  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ , where  $\pi$  and  $\sigma$  are permutations in  $S_n$ . In particular, we show that each of these properties may be used to characterize the Bruhat order on  $S_n$ .

## 1 Introduction

Let  $x = (x_{ij})$  be a generic square matrix and define  $\Delta_{I,I'}(x)$  to be the  $(I, I')$  minor of  $x$ , i.e., the determinant of the submatrix of  $x$  corresponding to rows  $I$  and columns  $I'$ . A real matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative. (See e.g. [9].) A polynomial  $p(x_{11}, \dots, x_{nn})$  in  $n^2$  variables is called totally nonnegative if it satisfies

$$p(A) \stackrel{\text{def}}{=} p(a_{1,1}, \dots, a_{n,n}) \geq 0 \quad (1)$$

for each  $n \times n$  totally nonnegative matrix  $A = (a_{i,j})$ . Some recent interest in total nonnegativity concerns a set of polynomials known in quantum Lie theory as the *dual canonical basis* of  $\mathcal{O}(SL(n, \mathbb{C}))$ . (See e.g. [25].) In particular, Lusztig [17] has proved that these polynomials are TNN.

A polynomial  $p(x)$  which is equal to a subtraction-free rational expression in matrix minors must be TNN. (By a result of Whitney [24], we need not be concerned that the denominator vanishes for some TNN matrices.) We shall say that such a polynomial  $p(x)$  has the *subtraction-free rational function* (SFR) property. If this subtraction-free rational

expression may be chosen so that the denominator is a monomial in matrix minors, we shall say that  $p(x)$  has the *subtraction-free Laurent* (SFL) property. One example of a polynomial having the SFL property is

$$x_{1,2}x_{2,1}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} + x_{1,3}x_{2,2}x_{3,1} \\ = \frac{\Delta_{13,23}(x)\Delta_{23,13}(x) + \Delta_{1,3}(x)\Delta_{3,1}(x)\Delta_{23,23}(x)}{\Delta_{3,3}(x)}.$$

Analogous classes of polynomials may be defined in terms of symmetric functions. (See [21, Ch. 7] for basic definitions concerning symmetric functions.) In particular, any finite submatrix of the infinite matrix  $H = (h_{j-i})_{i,j \geq 0}$ , where  $h_k$  is the  $k$ th complete homogeneous symmetric function and  $h_k = 0$  for  $k < 0$ , is called a *Jacobi-Trudi matrix*. We define a polynomial  $p(x_{1,1}, \dots, x_{n,n})$  to be *monomial nonnegative* (MNN) if for each Jacobi-Trudi matrix  $A = (a_{i,j})$  the symmetric function  $p(A)$  is equal to a nonnegative linear combination of monomial symmetric functions. Defining *Schur nonnegative* (SNN) polynomials analogously, we have that every SNN polynomial is MNN. Some recent interest in SNN polynomials is motivated by problems in algebraic geometry [8, Conj. 2.8, Conj. 5.1], [1].

## 2 Main result

The five nonnegativity properties defined in Section 1 have been applied most often to *immanants*, polynomials which belong to  $\text{span}_{\mathbb{C}}\{x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \mid \sigma \in S_n\}$ . (See [11], [12], [13], [20], [19], [22], [23]. The results of [7] may also be stated in these terms.) Curiously, the TNN, MNN, and SNN properties coincide when applied to immanants in the main theorems of the above papers. It is also curious that none of these immanants is known *not* to have the SFL property. It would be interesting to identify immanants which have some of these nonnegativity properties and fail to have others. Nevertheless, our main result shows that the five properties coincide when applied to immanants of the form

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

We shall use the following well-known characterizations of the Bruhat order on  $S_n$ . The Bruhat order on  $S_n$  is often defined by comparing two permutations  $\pi = \pi(1) \cdots \pi(n)$  and  $\sigma = \sigma(1) \cdots \sigma(n)$  according to the following criterion:  $\pi \leq \sigma$  if  $\sigma$  is obtainable from  $\pi$  by a sequence of transpositions  $(i, j)$  where  $i < j$  and  $i$  appears to the left of  $j$  in  $\pi$ . (See e.g. [14, p. 119].) A second well-known criterion compares permutations in terms of their defining matrices. Let  $M(\pi)$  be the matrix whose  $(i, j)$  entry is 1 if  $j = \pi(i)$  and zero otherwise. Defining  $[i] = \{1, \dots, i\}$ , and denoting the submatrix of  $M(\pi)$  corresponding to rows  $I$  and columns  $J$  by  $M(\pi)_{I,J}$ , we have the following.

**Theorem 1** *Let  $\pi$  and  $\sigma$  be two permutations in  $S_n$ . Then  $\pi$  is less than or equal to  $\sigma$  in the Bruhat order if and only if for all  $1 \leq i, j \leq n - 1$ , the number of ones in  $M(\pi)_{[i],[j]}$  is greater than or equal to the number of ones in  $M(\sigma)_{[i],[j]}$ .*

(See [2], [3], [4], [6], [10, pp. 173-177], [16], [15], [18]. for more characterizations.)

Our result, combined with those of our previous paper [5], is the following list of nonnegativity criteria with which one may define the Bruhat order.

**Theorem 2** *Let  $\pi$  and  $\sigma$  be permutations in  $S_n$ . The following conditions on  $\pi$  and  $\sigma$  are equivalent.*

1.  $\pi \leq \sigma$  in the Bruhat order.
2.  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  is totally nonnegative.
3.  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  is Schur nonnegative.
4.  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  is monomial nonnegative.
5.  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  has the subtraction-free rational function property.
6.  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  has the subtraction-free Laurent property.

*Proof:* The implications (3  $\Rightarrow$  4) and (6  $\Rightarrow$  5  $\Rightarrow$  2) are immediate. The implication (2  $\Rightarrow$  1) was established in [5, Thm. 2], and the implication (1  $\Rightarrow$  6) follows trivially from that proof. The implication (1  $\Rightarrow$  3) was established in [5, Thm. 3]. It will suffice therefore to prove the implication (4  $\Rightarrow$  1).

Suppose that  $\pi$  is not less than or equal to  $\sigma$  in the Bruhat order. By Theorem 1 we may choose indices  $1 \leq k, \ell \leq n-1$  such that  $M(\sigma)_{[k],[\ell]}$  contains  $q+1$  ones and  $M(\pi)_{[k],[\ell]}$  contains  $q$  ones. Keeping  $n$  fixed, let  $b$  be a large nonnegative integer which satisfies

$$\binom{2b}{b} > (2b + 2n)^{2n^2},$$

(which is possible because  $\binom{2b}{b}$  grows exponentially) and consider the  $n \times n$  Jacobi-Trudi matrix

$$B = \begin{bmatrix} h_{b+k-1} & \cdots & h_{b+k+\ell-2} & h_{2b+k-1} & \cdots & h_{2b+n+k-\ell-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_b & \cdots & h_{b+\ell-1} & h_{2b} & \cdots & h_{2b+n-1-\ell} \\ h_{n-k-1} & \cdots & h_{n-k+\ell-2} & h_{b+n-k-1} & \cdots & h_{b+2n-k-\ell-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_0 & \cdots & h_{\ell-1} & h_b & \cdots & h_{b+n-\ell-1} \end{bmatrix},$$

defined by the skew shape  $(2b+k-\ell-1)^k(b+n-\ell-1)^{n-k}/(b-\ell)^\ell$ . Let

$$s = k(2b+k-\ell-1) + (n-k)(b+n-\ell-1) - \ell(b-\ell)$$

be the number of boxes in this skew shape.

The polynomial  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  applied to  $B$  may be expressed as  $h_\lambda - h_\mu$  for some appropriate partitions  $\lambda, \mu$  of  $s$ , which depend on  $\pi, \sigma$ , respectively. We claim that the coefficient of  $m_1^s$  in the monomial expansion of  $h_\lambda - h_\mu$  is negative.

Note that the ratio of the coefficients of  $m_{1^s}$  in the monomial expansions of  $h_\lambda$  and  $h_\mu$  is

$$\frac{\binom{s}{\lambda_1, \dots, \lambda_n}}{\binom{s}{\mu_1, \dots, \mu_n}} = \frac{\mu_1! \cdots \mu_n!}{\lambda_1! \cdots \lambda_n!}.$$

By the locations of ones in the matrices  $M(\pi)$  and  $M(\sigma)$ , this ratio is less than or equal to

$$\frac{(2b+2n)^{k-q-1}}{(2b)^{k-q}} \frac{(b+2n)^{n-k-\ell+2q+2}}{b^{n-k-\ell+2q}} \frac{(2n)^{\ell-q-1}}{0^{\ell-q}},$$

which in turn is less than or equal to

$$\begin{aligned} \frac{(2b+2n)^{2n(k-q-1)}}{(2b)!} (b+2n)! (2b+2n)^{2n(n-k+q-1)} &= \frac{(b+2n)!^2}{(2b)!} (2b+2n)^{2n(n-2)} \\ &\leq \frac{(2b+2n)^{2n^2}}{\binom{2b}{b}}, \end{aligned}$$

which is less than 1 by our choice of  $b$ . It follows that the coefficient of  $m_{1^s}$  in the monomial expansion of  $h_\lambda - h_\mu$  is negative and the polynomial  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  is not MNN.  $\square$

## Acknowledgements

The authors are grateful to Arun Ram and an anonymous referee for helpful conversations.

## References

- [1] F. BERGERON, R. BIAGIOLI, AND M. ROSAS. Inequalities between Littlewood-Richardson coefficients (2004). Preprint math.CO/0403541 on ArXiv.
- [2] A. BJÖRNER. Orderings of Coxeter groups. In *Combinatorics and Algebra* (C. GREENE, ed.), vol. 34 of *Contemp. Math.*. American Mathematical Society, Providence, RI (1984), pp. 175–195.
- [3] A. BJÖRNER AND F. BRENTI. An improved tableau criterion for Bruhat order. *Electron. J. Combin.*, **3**, 1 (1996). Research paper 22, 5 pp. (electronic).
- [4] V. DEODHAR. Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. *Inventiones Math.*, **39** (1977) pp. 187–198.
- [5] B. DRAKE, S. GERRISH, AND M. SKANDERA. Two new criteria for comparison in the Bruhat order. *Electron. J. Combin.*, **11**, 1 (2004). Note 6, 4 pp. (electronic).
- [6] C. EHRESMANN. Sur la topologie de certains espaces homogènes. *Ann. Math.*, **35** (1934) pp. 187–198.
- [7] S. M. FALLAT, M. I. GEKHTMAN, AND C. R. JOHNSON. Multiplicative principal-minor inequalities for totally nonnegative matrices. *Adv. Appl. Math.*, **30**, 3 (2003) pp. 442–470.

- [8] S. FOMIN, W. FULTON, C. K. LI, AND Y. POON. Eigenvalues, singular values, and Littlewood-Richardson coefficients. *Amer. J. Math.*, **127**, 1 (2005) pp. 101–127.
- [9] S. FOMIN AND A. ZELEVINSKY. Total positivity: Tests and parametrizations. *Math. Intelligencer*, **22**, 1 (2000) pp. 23–33.
- [10] W. FULTON. *Young Tableaux; With Applications to Representation Theory and Geometry*, vol. 35 of *London Mathematical Society Student Texts*. Cambridge University Press, New York (1997).
- [11] I. P. GOULDEN AND D. M. JACKSON. Immanants of combinatorial matrices. *J. Algebra*, **148** (1992) pp. 305–324.
- [12] C. GREENE. Proof of a conjecture on immanants of the Jacobi-Trudi matrix. *Linear Algebra Appl.*, **171** (1992) pp. 65–79.
- [13] M. HAIMAN. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6**, 3 (1993) pp. 569–595.
- [14] J. E. HUMPHREYS. *Reflection groups and Coxeter groups*. Cambridge University Press (1990).
- [15] A. LASCOUX. Potentiel Yin sur le groupe symétrique. *Sém. Lothar. Combin.*, **38** (1996). Art. B38a, 12 pp. (electronic).
- [16] A. LASCOUX AND M. P. SCHÜTZENBERGER. Treillis et bases des groupes de Coxeter. *Electron. J. Combin.*, **3**, 2 (1996). Research paper 27, 35 pp. (electronic).
- [17] G. LUSZTIG. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston (1994), pp. 531–568.
- [18] R. A. PROCTOR. Classical Bruhat orders and lexicographic shellability. *J. Algebra*, **77** (1982) pp. 104–126.
- [19] B. RHOADES AND M. SKANDERA. Kazhdan-Lusztig immanants and products of matrix minors (2005). In progress.
- [20] B. RHOADES AND M. SKANDERA. Temperley-Lieb immanants. *Ann. Comb.* (2005). To appear.
- [21] R. STANLEY. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge (1999).
- [22] R. STANLEY AND J. R. STEMBRIDGE. On immanants of Jacobi-Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.
- [23] J. STEMBRIDGE. Some conjectures for immanants. *Can. J. Math.*, **44**, 5 (1992) pp. 1079–1099.
- [24] A. WHITNEY. A reduction theorem for totally positive matrices. *J. Anal. Math.*, **2** (1952) pp. 88–92.
- [25] A. ZELEVINSKY. From Littlewood-Richardson coefficients to cluster algebras in three lectures. In *Symmetric Functions 2001: Surveys of Developments and Perspectives* (S. FOMIN, ed.), vol. 74 of *NATO Science Series II: Mathematics, Physics, and Chemistry*. Kluwer, Dordrecht (2002), pp. 253–273.