Two New Criteria for Comparison in the Bruhat Order

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Abstract

We give two new criteria by which pairs of permutations may be compared in defining the Bruhat order (of type $A$). One criterion utilizes totally nonnegative polynomials and the other utilizes Schur functions.

The Bruhat order on $S_n$ is often defined by comparing permutations $\pi = \pi(1) \cdots \pi(n)$ and $\sigma = \sigma(1) \cdots \sigma(n)$ according to the following criterion: $\pi \leq \sigma$ if $\sigma$ is obtainable from $\pi$ by a sequence of transpositions $(i, j)$ where $i < j$ and $i$ appears to the left of $j$ in $\pi$. (See e.g. [7, p. 119].) A second well-known criterion compares permutations in terms of their defining matrices. Let $M(\pi)$ be the matrix whose $(i, j)$ entry is 1 if $j = \pi(i)$ and zero otherwise. Defining $[i] = \{1, \ldots, i\}$, and denoting the submatrix of $M(\pi)$ corresponding to rows $I$ and columns $J$ by $M(\pi)_{I,J}$, we have the following.

Theorem 1 Let $\pi$ and $\sigma$ be permutations in $S_n$. Then $\pi$ is less than or equal to $\sigma$ in the Bruhat order if and only if for all $1 \leq i, j \leq n - 1$, the number of ones in $M(\pi)_{[i],[j]}$ is greater than or equal to the number of ones in $M(\sigma)_{[i],[j]}$.

(See [1], [2], [3], [6, pp. 173-177], [8] for more criteria.) Using Theorem 1 and our defining criterion we will state and prove the validity of two more criteria.

Our first new criterion defines the Bruhat order in terms of totally nonnegative polynomials. A matrix $A$ is called totally nonnegative (TNN) if the determinant of each square submatrix of $A$ is nonnegative. (See e.g. [5].) A polynomial in $n^2$ variables $f(x_{1,1}, \ldots, x_{n,n})$ is called totally nonnegative (TNN) if for each TNN matrix $A = (a_{i,j})$
the number $f(a_{1,1}, \ldots, a_{n,n})$ is nonnegative. Some recent interest in TNN polynomials is motivated by problems in the study of canonical bases. (See [10,].)

**Theorem 2** Let $\pi$ and $\sigma$ be two permutations in $S_n$. Then $\pi$ is less than or equal to $\sigma$ in the Bruhat order if and only if the polynomial

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

is totally nonnegative.

**Proof:** ($\Rightarrow$) If $\pi = \sigma$ then (1) is obviously TNN. Suppose that $\pi$ is less than $\sigma$ in the Bruhat order. If $\pi$ differs from $\sigma$ by a single transposition $(i, j)$ with $i < j$, then we have $\pi(i) = \sigma(j) < \pi(j) = \sigma(i)$, and the polynomial (1) is equal to

$$\frac{x_{1,\pi(1)} \cdots x_{n,\pi(n)}}{x_{i,\pi(i)}x_{j,\pi(j)}} (x_{i,\pi(i)}x_{j,\pi(j)} - x_{i,\pi(j)}x_{j,\pi(i)})$$

which is clearly TNN. If $\pi$ differs from $\sigma$ by a sequence of transpositions, then the polynomial (1) is equal to a sum of polynomials of the form (2) and again is TNN.

($\Leftarrow$) Suppose that $\pi$ is not less than or equal to $\sigma$ in the Bruhat order. By Theorem 1 we may choose indices $1 \leq k, \ell \leq n - 1$ such that $M(\sigma)[k][\ell]$ contains $q + 1$ ones and $M(\pi)[k][\ell]$ contains $q$ ones. Now define the matrix $A = (a_{i,j})$ by

$$a_{i,j} = \begin{cases} 2 & \text{if } i \leq k \text{ and } j \leq \ell, \\ 1 & \text{otherwise}. \end{cases}$$

It is easy to see that $A$ is TNN, since all square submatrices of $A$ have determinant equal to 0, 1, or 2. Applying the polynomial (1) to $A$ we have

$$a_{1,\pi(1)} \cdots a_{n,\pi(n)} - a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = -2^q,$$

and the polynomial (1) is not TNN. □

Our second new criterion defines the Bruhat order in terms of Schur functions. (See [9, Ch. 7] for definitions.) Any finite submatrix of the infinite matrix $H = (h_{j-i})_{i,j \geq 0}$, where $h_k$ is the $k$th complete homogeneous symmetric function and $h_k = 0$ for $k < 0$, is called a Jacobi-Trudi matrix. Let us define a polynomial in $n^2$ variables $f(x_{1,1}, \ldots, x_{n,n})$ to be Schur nonnegative (SNN) if for each Jacobi-Trudi matrix $A = (a_{i,j})$ the symmetric function $f(a_{1,1}, \ldots, a_{n,n})$ is equal to a nonnegative linear combination of Schur functions. Some recent interest in SNN polynomials is motivated by problems in algebraic geometry [4, Conj. 2.8, Conj. 5.1].

**Theorem 3** Let $\pi$ and $\sigma$ be permutations in $S_n$. Then $\pi$ is less than or equal to $\sigma$ in the Bruhat order if and only if the polynomial

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

is Schur nonnegative.
Proof: \((\Rightarrow)\) If \(\pi = \sigma\) then \((3)\) is obviously SNN. Let \(A\) be an \(n \times n\) Jacobi-Trudi matrix and suppose that \(\pi\) is less than \(\sigma\) in the Bruhat order. If \(\pi\) differs from \(\sigma\) by a single transposition \((i, j)\), then for some partition \(\nu\) and some \(k, \ell, m\) \((\ell, m > 0)\), the evaluation of the polynomial \((3)\) at \(A\) is equal to

\[
h_\nu(h_{k+\ell}h_{k+m} - h_{k+\ell+m}h_k),
\]

and \((3)\) is clearly SNN. If \(\pi\) differs from \(\sigma\) by a sequence of transpositions, then the evaluation of \((3)\) at \(A\) is equal to a sum of polynomials of the form \((4)\) and again \((3)\) is SNN.

\((\Leftarrow)\) Suppose that \(\pi\) is not less than or equal to \(\sigma\) in the Bruhat order. By Theorem 1 we may choose indices \(1 \leq k, \ell \leq n - 1\) such that \(M(\sigma)[k],[\ell]\) contains \(q + 1\) ones and \(M(\pi)[k],[\ell]\) contains \(q\) ones. Now define the nonnegative number \(r = (k - q)(n + k - \ell - 2)\) and consider the Jacobi-Trudi matrix \(B\) defined by the skew shape \((n - 1 + 2r)^k(n - 1 + r)^{n-k}/r^\ell\),

\[
B = \begin{bmatrix}
h_{n-1+r} & \cdots & h_{n+\ell-2+r} & h_{n+\ell-1+2r} & \cdots & h_{2n-2+2r} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
h_{n-k+r} & \cdots & h_{n-k+\ell+1+r} & h_{n-k+\ell+2r} & \cdots & h_{2n-k-1+2r} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
h_{n-k-1} & \cdots & h_{n-k+\ell-2} & h_{n-k+\ell-1+2r} & \cdots & h_{2n-k-2+2r} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
h_0 & \cdots & h_{\ell-1} & h_{\ell+r} & \cdots & h_{n-1+r}
\end{bmatrix}.
\]

The polynomial \((3)\) applied to \(B\) may be expressed as \(h_\lambda - h_\mu\) for some appropriate partitions \(\lambda, \mu\) depending on \(\pi, \sigma\), respectively. We claim that \(\lambda\) is incomparable to or greater than \(\mu\) in the dominance order. Since \(M(\pi)[k],[\ell+1,n]\) contains \(k - q\) ones we have that

\[
\lambda_1 + \cdots + \lambda_{k-q} \geq (k - q)(n - k + \ell + 2r).
\]

Similarly, we have

\[
\mu_1 + \cdots + \mu_{k-q} \leq (k - q - 1)(2n - 2 + 2r) + \max\{n + \ell - 2 + r, 2n - k - 2 + r\}.
\]

Subtracting \((6)\) from \((5)\), we obtain

\[
(\lambda_1 + \cdots + \lambda_{k-q}) - (\mu_1 + \cdots + \mu_{k-q}) \geq n - \max\{\ell, n - k\} > 0,
\]

as desired.

Recall that the Schur expansion of \(h_\mu\) is

\[
h_\mu = s_\mu + \sum_{\nu \geq \mu} K_{\nu,\mu} s_\nu,
\]

where the comparison of partitions \(\nu > \mu\) is in the dominance order and the nonnegative Kostka numbers \(K_{\nu,\mu}\) count semistandard Young tableaux of shape \(\nu\) and content \(\mu\). (See e.g. [9, Prop. 7.10.5, Cor. 7.12.4].) It follows that the coefficient of \(s_\mu\) in the Schur expansion of \(h_\lambda - h_\mu\) is \(-1\) and the polynomial \((3)\) is not SNN. \(\square\)

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References


