1993

Decay of the Relative Error in the Formation of Acoustic Bullets

Harry E. Moses
Boston University

Reese T. Prosser
Dartmouth College

Decay of the Relative Error in the Formation of Acoustic Bullets

Harry E. Moses and Reese T. Prosser

Abstract

In a previous paper the authors have shown how to construct certain solutions of the acoustic and electromagnetic wave equations in three dimensions which are constrained asymptotically to outgoing spherical shells supported in narrow cones, i.e., which behave like “bullets”. In this report they show that in the acoustic case the magnitude of the relative error between the true solution and its asymptotic form decays in time according to an inverse square root law.

In [1] we have shown how to construct certain solutions of the acoustic and elecromagnetic wave equations in three dimensions which are constrained asymptotically to outgoing shells supported in narrow cones, i.e., which behave like “bullets”. In this report we show that in the acoustic case the magnitude of the relative error between the true solution and its asymptotic form satisfies the decay law

$$E(r,\theta,t) = \frac{f(r,\theta,t) - f_{\text{asy}}(r,\theta,t)}{1/r} = O\Big(\frac{1}{\sqrt{a+ct}}\Big).$$

1. Decay of the Relative Error

We use the notation conventions of reference [1] throughout. In particular,

$$f_{\text{asy}}(r,\theta,t) = (G/r)\eta[\sigma-\theta]\big\{\eta[r-a-ct] - \eta[r-b-ct]\big\},$$

where $0 < \sigma < \pi/2$ is the cone angle, and

$$f(r,\theta,t) = g_a(r,\theta,t) - g_b(r,\theta,t),$$

where

$$\begin{aligned} g_a(r,\theta,t) &= (G/r)\eta[\sigma-\theta]\big\{\eta[r-a-ct] - \eta[r\cos(\sigma-\theta)-a-ct]\big\} \\ &\quad + (G/r)(\nu_a/\pi)\big\{\eta[r\cos(\sigma-\theta)-a-ct] - \eta[r\cos(\sigma+\theta)-a-ct]\big\}, \end{aligned}$$

and similarly for $g_b(r,\theta,t)$. Here

$$\nu_a = \cos^{-1}\Big(\frac{\cos\sigma - \cos\beta_a\cos\theta}{\sin\beta_a\sin\theta}\Big), \qquad (0<\nu_a<\pi),$$

$$\beta_a = \cos^{-1}\Big(\frac{a+ct}{r}\Big), \qquad (0<\beta_a<\pi/2).$$

We are concerned with bounding the relative error

$$\begin{aligned} E(r,\theta,t) &= \frac{f(r,\theta,t) - f_{\rm asy}(r,\theta,t)}{(G/r)} \\ &= \big(\frac{\nu_a}{\pi}\big)\big\{\eta[r\cos(\sigma-\theta)-a-ct] - \eta[r\cos(\sigma+\theta)-a-ct]\big\} \\ &\quad - \big(\frac{\nu_b}{\pi}\big)\big\{\eta[r\cos(\sigma-\theta)-b-ct] - \eta[r\cos(\sigma+\theta)-b-ct]\big\} \\ &\quad - \eta[\sigma-\theta]\big\{\eta[r\cos(\sigma-\theta)-a-ct] - \eta[r\cos(\sigma-\theta)-b-ct]\big\} \end{aligned}$$

We see immediately that the support of this error term is a cylindrical wedge (See Figure 1), and that within this wedge the error term consists essentially of a difference of two arccosine terms which for large t are nearly equal (See Figures 2 and 3). Our goal now is to see how this error term decays as $t\to\infty$.

We consider separately the following cases (See Figure 1):

Case 1. Inside the cone ($0<\theta<\sigma$). In this case we suppose first that the point (r,θ) lies in the interior of the support wedge (See Figure 1):

$$\frac{b+ct}{\cos(\sigma-\theta)} < r < \frac{a+ct}{\cos(\sigma+\theta)}$$

In this region the relative error reduces to

$$E(r,\theta,t) = \frac{\nu_a-\nu_b}{\pi} < 0.$$

Here

$$\begin{aligned} \nu_a &= \cos^{-1} u_a, \\ u_a &= \frac{\cos\sigma - \cos\beta_a\cos\theta}{\sin\beta_a\sin\theta}, \\ \beta_a &= \cos^{-1} v_a, \\ v_a &= \frac{a+ct}{r}, \end{aligned}$$

and similarly for ν_b. Since r is large when t is large, and since

$$v_b - v_a = \frac{b-a}{r} > 0$$

is small when r is large, we can then safely approximate the difference $\nu_b - \nu_a > 0$ by the differential $d\nu_a$:

$$\begin{aligned}
\nu_b - \nu_a &\cong \frac{1}{\sqrt{1-u_a^2}}\ du_a, \\
&= \Big(\frac{1}{\sqrt{1-u_a^2}}\Big)(\cot\theta - u_a \cot\beta_a)\ d\beta_a, \\
&= \Big(\frac{1}{\sqrt{1-u_a^2}}\Big)(\cot\theta - u_a \cot\beta_a)\Big(\frac{1}{\sqrt{1-v_a^2}}\Big)\ \frac{b-a}{r}.
\end{aligned}$$

In this region we have $v_a < 1$ and $u_a < 1$, so when t, and hence r, is large, we have

$$|E(r,\theta,t)| < C/r,$$

where the constant C depends on a, b, σ, θ, but not on r or t.

We suppose next that the point (r,θ) lies near the leading edge of the support wedge (See Figure 1):

$$\frac{a+ct}{\cos(\sigma+\theta)} < r < \frac{b+ct}{\cos(\sigma+\theta)}$$

In this region the relative error is

$$E(r,\theta,t) = \frac{-\nu_b}{\pi} < 0.$$

In this region we find that $u_b = 1$ when $r = (b+ct)/\cos(\sigma+\theta)$, so the estimates in the previous region no longer hold. Instead we consider the differential $d(\nu_b^2)$:

$$\begin{aligned}
d(\nu_b^2) &= 2\nu_b\ d\nu_b \\
&= \Big(\frac{2\nu_b}{\sqrt{1-u_b^2}}\Big)(\cot\theta - u_b \cot\beta_b)\Big(\frac{1}{\sqrt{1-v_b^2}}\Big)\ dv_b.
\end{aligned}$$

When r is near $(b+ct)/\cos(\sigma+\theta)$, then u_b is near 1 and so ν_b is near 0. The ratio $2\nu_b/\sqrt{1-u_b^2}$ is then, by l"Hôpital's Rule, near

$$\lim_{u\to 1}\frac{2\cos^{-1}u}{\sqrt{1-u^2}} = \lim_{u\to 1}\frac{2}{u} = 2.$$

Now when t, and hence r, is large, then $v_b - v_a = (b-a)/r$ is small, and

$$(\nu_b - 0)^2 \cong 2(\cot\theta - u_b\cot\beta_b)\Big(\frac{1}{\sqrt{1-v_b^2}}\Big)\ \frac{b-a}{r}.$$

It follows that in this region we have

$$|E(r,\theta,t)| < C/\sqrt{r}.$$

A similar argument holds when we suppose that (r,θ) lies near the trailing edge of the support wedge (See Figure 1):

$$\frac{a+ct}{\cos(\sigma-\theta)} < r < \frac{2b-a+ct}{\cos(\sigma-\theta)}.$$

Then the relative error is

$$E(r,\theta,t) = \frac{\nu_a-\pi}{\pi} < 0.$$

In this region we use the differential relation

$$d(\pi-\nu_a)^2 = \Big(\frac{2(\pi-\nu_a)}{\sqrt{1-u_a^2}}\Big)(\cot\theta - u_a\cot\beta_a)\Big(\frac{1}{\sqrt{1-v_a^2}}\Big)\ dv_a.$$

When r is large, then u_a is near -1 and ν_a is near π. Hence the ratio

$$\frac{2(\pi-\nu_a)}{\sqrt{1-u_a^2}} \to \lim_{u\to -1}\frac{-2}{u} = 2,$$

and so

$$(\pi-\nu_a)^2 \cong 2(\cot\theta - u_a\cot\beta_a)\Big(\frac{1}{\sqrt{1-v_a^2}}\Big)\frac{b-a}{r}.$$

It follows that in this region we also have

$$|E(r,\theta,t)| < C/\sqrt{r}.$$

We note here that when t is small and θ is small, then the three regions considered above are no longer distinct, and the arguments given above are then invalid (See Figure 1). More precisely, these three regions are distinct only outside the cylinder of radius $\rho = (b-a)/2\tan\sigma$ about the z–axis. On the edge of this cylinder we have $r\sin\theta = \rho$. If θ remains fixed as t increases, however, the three regions eventually separate and the arguments given above then apply, i.e, they apply for all sufficiently large t.

We also note that when θ is small, then $\cot\theta$ is large, and we lose the force of our estimate. Similarly, when θ is near σ, then β_a is near 0 and $\cot\beta_a$ is large, and again we lose the force of our estimate. If we restrict our attention to the region $0 < \theta_1 \leq \theta \leq \theta_2 < \sigma$, where θ_1 and θ_2 are arbitrary, then for large t this restricted region lies outside the excluded cylinder described above, and in this restricted region the estimates above are all uniform in θ.

Computer based calculations suggest that when $\theta = 0$ and $a + ct < r < b + ct$ the relative error term $E(r, \theta, t) = -1$ for all t, and so does not decay at all with time (See Figure 4). Similarly, when $\theta = \sigma$ and $a + ct < r < b + ct$ the relative error term $E(r, \theta, t) = -1/2$ at the trailing edge for all t, and also does not decay at all with time (See Figure 5). Hence the restriction on the support region described above is necessary for the decay law to hold.

Case 2. Outside the cone ($\sigma < \theta < \sigma + \pi/2$). Arguments similar to those used above in Case 1. now show that in Case 2., for the interior of the support wedge,

$$(b+ct)/\cos(\theta-\sigma) < r < (a+ct)/\cos(\theta+\sigma),$$

we have

$$E(r,\theta,t) = \frac{\nu_a - \nu_b}{\pi},$$

and

$$|E(r,\theta,t)| < C/r.$$

For the leading edge of the support wedge:

$$(2a-b+ct)/\cos(\theta+\sigma) < r < (b+ct)/\cos(\theta+\sigma),$$

we have

$$\frac{-\nu_b}{\pi} \leq E(r,\theta,t) \leq \frac{\nu_a-\nu_b}{\pi} < 0,$$

and

$$|E(r,\theta,t)| < C/\sqrt{r}.$$

For the trailing edge of the support wedge:

$$(a + ct/\cos(\theta - \sigma) < r < (b + ct)/\cos(\theta - \sigma),$$

we have

$$\frac{\nu_a - \nu_b}{\pi} \leq E(r, \theta, t) \leq \frac{\nu_a}{\pi},$$

and again

$$|E(r, \theta, t)| < C/\sqrt{r}.$$

These estimates are again uniform in θ away from $\theta = \sigma$, i.e., for $\sigma < \theta_3 \leq \theta \leq \pi/2$, for arbitrary θ_3.

We conclude that in the restricted regions $0 < \theta_1 \leq \theta \leq \theta_2 < \sigma$ and $\sigma < \theta_3 \leq \theta \leq \pi/2$ the relative error obeys for large t the decay law

$$|E(r, \theta, t)| < C/\sqrt{r},$$

with the constant depending only on $a, b, \theta_1, \theta_2, \theta_3$.

Finally, since $r > a + ct$ within the support wedge, we also have

$$|E(r, \theta, t)| < C/\sqrt{a + ct}$$

in the same region.

Bibliography

1. H. E. Moses and R.T.Prosser, *Acoustic and Electromagnetic Bullets*, SIAM J. Appl. Math. **50** (1990), pp. 1325-1340.