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Almost Avoiding Permutations

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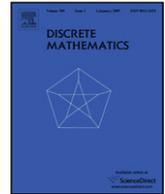
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Note

Almost avoiding permutations

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ABSTRACT

We investigate the notion of almost avoiding a permutation: π almost avoids β if one can remove a single entry from π to obtain a β -avoiding permutation.

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1. Introduction

The permutation π of length n , written in one-line notation as $\pi(1)\pi(2)\cdots\pi(n)$, is said to contain the permutation σ if π has a subsequence that is order isomorphic to σ , and each such subsequence is said to be an occurrence of σ in π or simply a σ pattern. For example, $\pi = 491867532$ contains $\sigma = 51342$ because of the subsequence $\pi(2)\pi(3)\pi(5)\pi(6)\pi(9) = 91672$. Permutation containment is easily seen to be a partial order on the set of all (finite) permutations, which we simply denote by \leq . If the permutation π fails to contain σ we say that π avoids σ .

A downset in this permutation containment order is referred to as a permutation class; in other words, if \mathcal{C} is a permutation class, $\pi \in \mathcal{C}$ and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$. We denote by \mathcal{C}_n the set $\mathcal{C} \cap \mathcal{S}_n$ (the permutations of length n in \mathcal{C}) and we refer to $\sum_{n \geq 0} |\mathcal{C}_n| x^n$ as the generating function of \mathcal{C} . Given any set of permutations B , the set $\text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$ forms a permutation class and, conversely, for any permutation class \mathcal{C} there is a unique antichain (set of pairwise incomparable elements) B such that $\mathcal{C} = \text{Av}(B)$; we call this antichain the basis of \mathcal{C} , and say that \mathcal{C} is finitely based if B is finite.

One area in which permutation classes arise is the study of sorting machines. For example, Knuth [7] showed that the class $\text{Av}(231)$ consists precisely of those permutations that can be sorted by a stack (a last-in first-out linear sorting machine), while Tarjan [12] observed that the class of permutations that can be sorted by a network consisting of two parallel queues (first-in first-out linear sorting machines) is $\text{Av}(321)$. A classic result in the field of permutation containment is that the number of permutations in $\text{Av}(231)$ and in $\text{Av}(321)$ of length n are both equal to the n th Catalan number. For bijections between the two sets, see the recent survey by Claesson and Kitaev [3].

Our interest in this paper is with permutations which “almost lie” in a given permutation class, a concept which we formalize as follows: given a permutation class \mathcal{C} and natural number t , we say that the permutation π t -almost lies in (or

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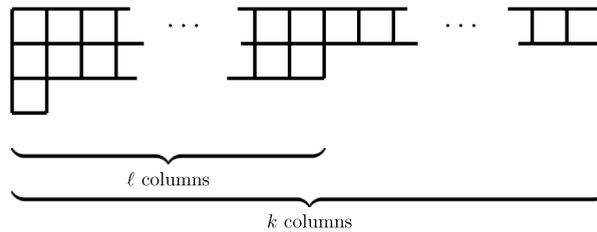


Fig. 1. The shape of a Standard Young Tableaux obtained from the Robinson–Schensted algorithm applied to a permutation in $\text{Av}(321)^{+1}$ that contains at least one 321 pattern.

simply *almost lies in* if $t = 1$) \mathcal{C} if one can remove t (or fewer) entries from π to obtain a permutation that lies in \mathcal{C} . We denote the set of permutations which t -almost lie in \mathcal{C} by \mathcal{C}^{+t} ; note that \mathcal{C}^{+t} is a permutation class itself.

The notion of almost avoiding permutations has a natural interpretation in terms of sorting machines: if \mathcal{C} consists of those permutations which can be sorted by the machine M then the permutations in \mathcal{C}^{+1} are those which can be sorted by M in parallel with a *one-time use buffer*, which we define as a machine which can hold one entry, once in the sorting process. The classes \mathcal{C}^{+t} for $t \geq 2$ then consist of those permutations which can be sorted by M in parallel with t one-time use buffers.

Note that almost avoidance classes differ from the classes introduced by Noonan [9], who studied permutations with at most one copy of 321; this permutation class, which we denote by $\text{Av}(321^{\leq 1})$, is strictly contained in $\text{Av}(321)^{+1}$.

In the following two sections we provide the enumeration of $\text{Av}(231)^{+1}$ and $\text{Av}(321)^{+1}$, i.e., the permutations that can be “almost stack-sorted” and “almost sorted by two parallel queues”, and then end with a conjecture. By the usual symmetries of permutations, this completes the enumeration of permutations which almost avoid a pattern of length 3. Before this, we conclude the introduction with a general result, first proved by the Theory of Computing Research Group at the University of Otago in 2002.

Proposition 1 (Otago Theory of Computing Research Group). *For any finitely based class \mathcal{C} and positive integer t , the class \mathcal{C}^{+t} is finitely based.*

Proof. It suffices to prove that \mathcal{C}^{+1} is finitely based if \mathcal{C} is, as then the proposition follows by iteration. Suppose that the longest basis element of \mathcal{C} has length m and consider a permutation $\tau \notin \mathcal{C}^{+1}$. Since $\tau \notin \mathcal{C}$, τ contains a subsequence of length at most m order isomorphic to a basis element of \mathcal{C} . Furthermore, since $\tau \notin \mathcal{C}^{+1}$, every time we remove a single element from this subsequence, we find another occurrence of a basis element of \mathcal{C} (which is also of length at most m). By taking the original subsequence together with these additional occurrences of basis elements we see that τ contains a permutation of length at most $m(m + 1)$ which does not lie in \mathcal{C}^{+1} , verifying that the basis of \mathcal{C}^{+1} is finite. \square

2. $\text{Av}(321)^{+1}$

To enumerate the permutations in $\text{Av}(321)^{+1}$ we use the Robinson–Schensted algorithm. While a detailed description of this algorithm can be found in Sagan’s text [10], a few details suffice for our arguments. First recall that the Robinson–Schensted algorithm associates to each permutation π of length n a pair, denoted $(P(\pi), Q(\pi))$, of standard Young tableaux (SYT), each with n cells and of the same shape. We denote the shape of $P(\pi)$ by $\text{sh } P(\pi)$, so in the case where π is of length n , $\text{sh } P(\pi) = \text{sh } Q(\pi)$ is a partition, say $\lambda = (\lambda_1, \dots, \lambda_r)$ of n (which we denote by $\lambda \vdash n$). Schensted [11] proved that the length of the longest decreasing subsequence π is equal to the number of rows of $P(\pi)$, and thus if $\text{sh } P(\pi) = \lambda = (\lambda_1, \dots, \lambda_r)$, then the longest decreasing subsequence of π is of length r . Greene [5] gave a generalization of Schensted’s theorem, from which the following proposition routinely follows.

Proposition 2. *If $\text{sh } P(\pi) = (\lambda_1, \dots, \lambda_r)$ then the longest $k \cdots 21$ -avoiding subpermutation in π has length $\lambda_1 + \dots + \lambda_{k-1}$.*

The first step in our enumeration is to characterize the shapes of SYT that can arise from a permutation which almost avoids 321.

Proposition 3. *The permutation π lies in $\text{Av}(321)^{+1}$ if and only if $\text{sh } P(\pi)$ is of the form (k) , (k, ℓ) , or $(k, \ell, 1)$ for some integers $k \geq \ell \geq 1$, see Fig. 1.*

Proof. No permutation in $\text{Av}(321)^{+1}$ can contain a 4321 pattern as then there would be no entry whose removal gives a 321-avoiding permutation. Similarly, no permutation in $\text{Av}(321)^{+1}$ can contain two disjoint occurrences of 321. This means that such a permutation cannot contain a 123-avoiding permutation of length 6. Translating into SYT, this means that, for all $\pi \in \text{Av}(321)^{+1}$, $\text{sh } P(\pi)$ has no columns of length 4 or greater (it avoids 4321) and at most one column of length 3 (its longest 123-avoiding subpermutation has length at most 5, so this follows from Proposition 2). This implies that $\text{sh } P(\pi)$ is of one of the forms listed in the statement of the proposition. For the other direction, note that if $\text{sh } P(\pi) = (k)$ or $\text{sh } P(\pi) = (k, \ell)$ then π avoids 321, while if $\text{sh } P(\pi) = (k, \ell, 1)$ then π contains a subpermutation of length $k + \ell$ which avoids 321, completing the proof. \square

Note that the proof of Proposition 3 shows that $\pi \in \text{Av}(321)^{+1}$ if and only if π avoids 4321 and does not contain two disjoint occurrences of 321. Thus the basis elements of $\text{Av}(321)^{+1}$ are all of length at most 6 and can be readily generated by computer.

By Proposition 3 and the Robinson–Schensted algorithm we now have that

$$|\text{Av}_n(321)^{+1}| = \sum_{\lambda=(k,\ell)\vdash n} (f^\lambda)^2 + \sum_{\lambda=(k,\ell,1)\vdash n} (f^\lambda)^2,$$

where f^λ denotes the number of SYT of shape λ . The first sum is simply the number of 321-avoiding permutations which, as stated in the introduction, is equal to the n th Catalan number, C_n . To evaluate the second sum we use the Hook Length Formula, which states that for $\lambda \vdash n$, f^λ is equal to $n!$ divided by the product of the hook lengths of cells in the Ferrers diagram of λ . In the case of $\lambda = (k, \ell, 1)$, the product of the hook lengths of cells in the top row is

$$(k + 2)k \cdots (k - \ell + 2)(k - \ell) \cdots 1 = \frac{(k + 2)k!}{(k - \ell + 1)},$$

the product for the middle row is $(\ell + 1)(\ell - 1)!$, and the solitary cell in the bottom row has a hook length of 1. Thus we have:

$$\begin{aligned} |\text{Av}_n(321)^{+1}| &= C_n + \sum_{(k,\ell,1)\vdash n} \left(\frac{n!(k - \ell + 1)}{(k + 2)k!(\ell + 1)(\ell - 1)!} \right)^2 \\ &= C_n + \sum_{k=\lfloor n/2 \rfloor}^{n-2} \left(\frac{n!(2k - n + 2)}{(k + 2)k!(n - k)(n - k - 2)!} \right)^2. \end{aligned}$$

An empirical calculation in Maple implies that these numbers likely have the generating function

$$\frac{1 - 8x + 13x^2 + 24x^3 - 48x^4 - (1 - 6x + x^2 + 34x^3 - 26x^4 - 4x^5)\sqrt{1 - 4x}}{2x^2(1 - x)(1 - 4x)^2}.$$

3. $\text{Av}(231)^{+1}$

Our approach to enumerating the class $\text{Av}(231)^{+1}$ differs significantly from the approach used in the previous section and makes use of the following definition: the entry $\pi(i)$ in π is called *essential* if its removal results in a 231-avoiding permutation. For example, the permutation $\pi = 1742653$ contains two essential entries, $\pi(5) = 4$ and $\pi(7) = 3$. (Also note that if $\pi \in \text{Av}(231)$, then by our definition every entry of π is essential.) As a first step, we make the following observation.

Proposition 4. *An essential entry in the permutation $\pi \in \text{Av}(231)^{+1}$ participates as the minimum entry in either all or none of the occurrences of 231 in π .*

Proof. Suppose, to the contrary, that $\pi \in \text{Av}(231)^{+1}$ contains indices $i < j < k$ such $\pi(i)\pi(j)\pi(k)$ is order isomorphic to 231 and $\pi(k)$ is essential, but that $\pi(k)$ also participates in another 231 pattern as a non-minimal element. Label the minimal element of this later pattern $\pi(\ell)$. Clearly $\pi(i)\pi(j)\pi(\ell)$ is also order isomorphic to 231, so $\pi - \pi(k) \notin \text{Av}(231)$, a contradiction to the assumption that $\pi(k)$ is essential. \square

By Proposition 4 we can divide the essential entries of a permutation $\pi \in \text{Av}(231)^{+1}$ into *small essential entries*, which participate as the minimum entry in all occurrences of 231, and *large essential entries*, which participate as the minimum entry in no occurrences of 231.

In enumerating $\text{Av}(231)^{+1}$, we use two generating functions frequently,

$$\begin{aligned} f &= \text{the generating function for } \text{Av}(231)^{+1}, \text{ and} \\ c &= \text{the generating function for the Catalan numbers, so also for } \text{Av}(231), \\ &= \frac{1 - \sqrt{1 - 4x}}{2x}. \end{aligned}$$

Both of these generating functions have constant term 1.

Proposition 5. *The generating function for permutations in $\text{Av}(231)^{+1} \setminus \text{Av}(231)$ in which the greatest element is not involved in a copy of 231 is given by $2(f - c)xc$.*

Proof. The plot of a permutation of the specified form can be divided into the greatest entry and two regions, A and B , as depicted in Fig. 2. One (but not both) of the two regions A or B must contain a permutation in $\text{Av}(231)^{+1} \setminus \text{Av}(231)$, while the other must contain a 231-avoiding permutation. This leads to the generating function specified in the proposition. \square

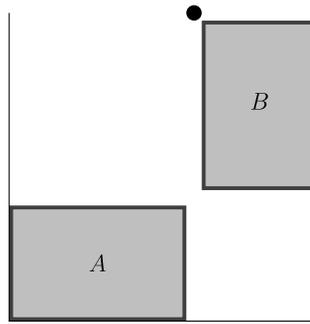


Fig. 2. A permutation in $Av(231)^{+1} \setminus Av(231)$ in which the greatest element is not involved in a copy of 231.

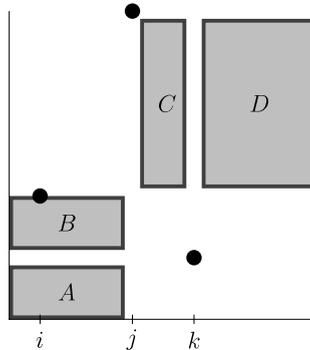


Fig. 3. A permutation with a small essential entry in which the greatest entry is involved in at least one copy of 231 but is not essential.

Proposition 6. *The generating function for permutations in $Av(231)^{+1}$ with an essential greatest (resp., leftmost, rightmost, or least) entry is $x^2c' + xc - c + 1$.*

Proof. The cases are all similar, so we count permutations with an essential greatest entry. To construct such a permutation, one must insert a new greatest entry into a 231-avoider (such permutations have the generating function $x^2c' + xc$ because there are $n + 1$ ways to insert a new greatest entry into any 231-avoider, and the number of 231-avoiders is the n th Catalan number) without creating a 231-avoider (these have the generating function $c - 1$). \square

For the following results we need a bit of notation; for a permutation $\pi \in S_n$ and sets $A, B \subseteq [n]$, we write $\pi(A \times B)$ for the permutation which is order isomorphic to the subsequence of π which has indices from A and values in B .

Proposition 7. *The generating function for permutations in $Av(231)^{+1}$ with an essential small entry in which the greatest entry participates in a copy of 231 but is not essential is $x^4c'^2 + xc(x^2c' + xc - c + 1)$.*

Proof. Take π of length n specifying the hypotheses, and suppose that $\pi(j) = n$ and that the small essential entry is $\pi(k)$. As n must be involved in at least one copy of 231, $\pi([1, j] \times (\pi(k), n))$ must be nonempty; let $\pi(i)$ denote the greatest entry in this region. By this choice of i , $\pi([1, j] \times (\pi(i), n))$ is empty, and because $\pi(k)$ is essential, $\pi((j, n] \times [1, \pi(i)))$ contains only the entry $\pi(k)$.

We therefore have three types of entries: the small essential $\pi(k)$, the entries in $\pi([1, j] \times [1, \pi(i)])$, and the entries in $\pi((j, n] \times (\pi(i), n))$. We further divide these latter regions into A, B, C and D , as indicated in Fig. 3.

As the entry n is not essential, one of two situations must occur:

- (S1) π has an entry in C , or
- (S2) $\pi(k)$ forms a copy of 231 with two entries from B .

Conversely, any permutation of this form that satisfies (S1) or (S2) is of the desired form.

First we count the permutations satisfying (S1). In this case the entries in $A \cup B$ form a 231-avoiding permutation, as do the entries in $C \cup D$; thus both sets of entries are counted by c . The generating function for arrangements of $\pi(k)$ among the entries in $A \cup B$ is therefore x^2c' , and this is the same as the generating function for arrangements of $\pi(k)$ among the entries in $C \cup D$. Multiplying these functions together double-counts the entry $\pi(k)$, but fails to count $\pi(n)$, so the total contribution of the permutations of this type satisfying (S1) is $x^4c'^2$.

Now we need to count the permutations that satisfy (S2) but not (S1). We know that π does not have an entry in C and that the entries of π in D avoid 231. Furthermore, $\pi(k)$ and the entries in $A \cup B$ form a permutation with an essential rightmost entry, so their generating function is $x^2c' + xc - c + 1$ by Proposition 6. After taking into account the contribution of n , permutations of this type contribute $xc(x^2c' + xc - c + 1)$, proving the proposition. \square

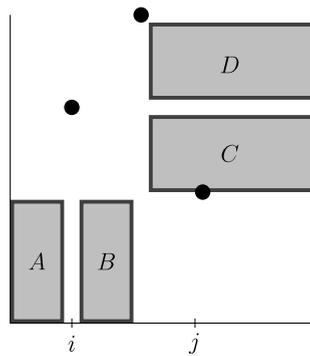


Fig. 4. A permutation with a large essential entry in which the greatest entry participates in a copy of 231 but is not essential.

Proposition 8. *The generating function for permutations in $\text{Av}(231)^{+1}$ without a small essential entry, with a large essential entry that is not the greatest entry, and in which the greatest entry participates in at least one copy of 231 is given by $x^2c'(x^2c' + xc - c + 1) + (x^2c' + x^2c - c + x + 1)(c - 1)$.*

Proof. By arguments analogous to the proof of Proposition 7 it can be established that these permutations are of the form depicted in Fig. 4 (in this figure the essential large entry is $\pi(i)$). Note that, as indicated by the figure, π must contain an entry in region C as otherwise the greatest element of π would not lie in a copy of 231. We divide these permutations into two types:

- (L1) there is a copy of 231 containing $\pi(i)$ and entries in $A \cup B$ or
- (L2) there is no such copy of 231.

In case (L1), the entries in $A \cup B$ must avoid 231 because $\pi(i)$ is essential, so they are counted by c . Thus x^2c' is the generating function for the number of arrangements of $\pi(i)$ together with the entries in $A \cup B$. We then have that $\pi(i)$ is a leftmost essential element of the permutation given by it and the entries in $C \cup D$, so Proposition 6 shows that these entries are counted by $x^2c' + xc - c + 1$. Multiplying these functions double-counts $\pi(i)$ but does not count the greatest entry of π , so the contribution of the (L1) permutations is $x^2c'(x^2c' + xc - c + 1)$.

In case (L2), $\pi(i)$ and the entries in $A \cup B$ form a nonempty 231-avoiding permutation, and are thus counted by $c - 1$. The entries in $C \cup D$ together with $\pi(i)$ form a permutation with an essential leftmost entry, but we cannot directly apply Proposition 6 because if π had a unique entry in C then that entry would be a small essential entry, and we do not wish to count such permutations. Thus we subtract the generating function for permutations with an essential leftmost entry of value 2; it is easily seen that this generating function is $x(c - xc - 1)$, and so the contribution of permutations in case (L2) is $(x^2c' + x^2c - c + x + 1)(c - 1)$, completing the proof. \square

Theorem 9. *The generating function for the permutations that almost avoid 231 is*

$$\frac{1 - 5x - 6x^2 + 45x^3 - 24x^4 - (1 + x - 4x^2 + x^3)(1 - 4x)^{3/2}}{-2x^2(1 - 4x)^{3/2}}.$$

Proof. Letting f denote the generating function for permutations that almost avoid 231, Propositions 5–8 show that

$$f = c + 2xc(f - c) + x^2c' + xc - c + 1 + x^4c'^2 + xc(x^2c' + xc - c + 1) + x^2c'(x^2c' + xc - c + 1) + (x^2c' + x^2c - c + x + 1)(c - 1),$$

from which the desired solution follows. \square

4. Open problems

Our computations indicate that $|\text{Av}_n(321)^{+1}| < |\text{Av}_n(231)^{+1}|$ for all $n \geq 4$, which begs for a combinatorial explanation:

Problem 10. Construct a length-preserving injection from $\text{Av}(321)^{+1}$ to $\text{Av}(231)^{+1}$.

Finally, we conclude with a conjecture about the exact enumeration problem.

Conjecture 11. *For all t , the generating functions for $\text{Av}(231)^{+t}$ and $\text{Av}(321)^{+t}$ are rational in x and $\sqrt{1 - 4x}$.*

We note that there has been similar work done for sets of permutations with at most t copies of these patterns. Bóna [1] proved that the generating function for $\text{Av}(231^{\leq t})$ is rational in x and $\sqrt{1 - 4x}$ (see also Mansour and Vainshtein [8] and Brignall, Huczynska, and Vatter [2]). For the pattern 321, Noonan [9] enumerated $\text{Av}(321^{\leq 1})$, while Fulmek [4] counted $\text{Av}(321^{\leq 2})$ and conjectured that the generating function for $\text{Av}(321^{\leq t})$ is rational in x and $\sqrt{1 - 4x}$.

The study of almost avoidance is extended to pairs of permutations of length 3 by Griffiths, Smith, and Warren [6].

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