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# Rotational Covariance and Greenberger-Horne-Zeilinger theorems for three or more particles of any dimension

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## Abstract

Greenberger-Horne-Zeilinger (GHZ) states are characterized by their transformation properties under a continuous symmetry group, and  $N$ -body operators that transform covariantly exhibit a wealth of GHZ contradictions. We show that local or noncontextual hidden variables cannot duplicate the predicted measurement outcomes for covariant transformations, and we extract specific GHZ contradictions from discrete subgroups, with no restrictions on particle number  $N$  or dimension  $d$  except for the general requirement that  $N \geq 3$  for GHZ states. However, the specific contradictions fall into three regimes distinguished by increasing demands on the number of measurement operators required for the proofs. The first regime consists of proofs found recently by Ryu et. al. [33], the first operator-based theorems for all odd dimensions,  $d$ , covering many (but not all) particle numbers  $N$  for each  $d$ . We introduce new methods of proof that define second and third regimes and produce new theorems that fill all remaining gaps down to  $N = 3$ , for every  $d$ . The common origin of all such GHZ contradictions is that the GHZ states and measurement operators transform according to different representations of the symmetry group, which has an intuitive physical interpretation.

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## I. INTRODUCTION

The groundbreaking discovery by Greenberger, Horne and Zeilinger [1, 2] of nonprobabilistic contradictions [3] between quantum predictions and local hidden variables theories (now called GHZ contradictions), and Mermin's demonstration [4] that these embody Kochen-Specker contextuality [5, 6] as well as Bell-EPR nonlocality [7], sparked vigorous developments along a number of lines. These included GHZ theorems for many qubits [8, 9], improved Bell [10] and Kochen-Specker theorems [11–13], Bell theorems for two particles of arbitrary dimension  $d$  [14–16], and the introduction of noncontextuality inequalities, state dependent [17] and state-independent [18, 19]. Connections of GHZ theorems to practical pursuits of quantum cryptography [20, 21] and quantum error correction [22] have been made. An error correction protocol employing concatenated GHZ states in particular was recently proposed [23].

Of particular interest here is the extension of GHZ contradictions to systems of both higher dimension  $d$  and a broader range of particle numbers  $N$ . Zukowski and Kaszlikowski [24] described an experimental protocol involving spatially separated arrays of beam splitters, phase shifters, and detectors that would show GHZ contradictions for  $N$  particles, each of dimension  $d = N - 1$ . The same authors [25] found a similar protocol for  $N$  particles, each of dimension  $d = N$ , but with only probabilistic quantum predictions. Cerf, Massar, and Pironio [26] found GHZ theorems in the form of Kochen-Specker operator identities, based on a compatible set of Pauli operators (stabilizers) analogous to Mermin's, for all even dimensions  $d$  and all odd  $N \geq d + 1$ . They also established criteria for a contradiction to be genuinely (or irreducibly) multiparty ( $N$ ) or multidimensional ( $d$ ). Lee, Lee, and Kim [27] extended these results to include *all* odd  $N \geq 3$ , for every even  $d$ . They accomplished this by using concurrent operators - operators which have a common eigenstate even if they do not commute. They realized that a common eigenstate allows the establishment of EPR elements of reality [28] and thus GHZ contradictions, although noncommutativity makes these contradictions state-dependent and rules out KS operator identities. Recently, Tang et. al. further extended these results by deriving contradictions for all even  $N \geq 4$ , for all even  $d$ . The  $d \geq 4$  proofs were based on stabilizers of GHZ graph states, allowing state-independent contextuality inequalities [29], while the more challenging  $d = 2$  proofs (with  $N$  even) used concurrent operators with more than two measurement settings for each

particle [30]. Waegell and Aravind [31] have explored systematically both observable-based and projector-based proofs of the KS theorem, based on the  $N$ -qubit Pauli group for all numbers of qubits  $N \geq 2$ . Such proofs number in the thousands, and a subset can be converted into GHZ paradoxes [32] for all even  $N \geq 4$ . Very recently, Ryu et. al. [33, 34] found GHZ theorems for all  $d$ , with infinite sequences of  $N$  for each  $d$ , answering a long-standing question whether odd- $d$  contradictions could be found beyond the line  $N = d + 1$  of Ref. [24]. They did this by extending the concurrent operator approach introduced in Ref. [27], so that these contradictions are also state-dependent. This result sharpened the question whether state-independent contradictions could be found for odd  $d$  using compatible sets of Pauli operators. While deriving state- $d$ dependent contextuality proofs for such cases, Howard et. al. [35] answered the above question in the negative, arguing that previous results of Gross and of Veitch et. al. [36] rule out state-independent proofs with stabilizer measurements of any odd  $d$ .

Also of interest for this work is the general connection of symmetry to GHZ contradictions. It is noteworthy that the seminal GHZ paper [1] used the rotational symmetry of a four-particle state to derive contradictions; it was noted in passing that three particles would suffice. The only other derivations that make explicit use of GHZ state symmetry, as far as I am aware, are the concurrent operator approaches [27, 30, 33, 34]. There, tensor product operators are found that preserve the GHZ state while transforming a particular  $N$ -body operator into others that form a concurrent set.

In this work we suggest a broader role for symmetry arguments in deriving and interpreting GHZ contradictions. With GHZ states providing an appropriate first case, we describe an  $N$ -body uniaxial rotation group that characterizes GHZ entanglement and generates continuous sets of covariant  $N$ -body operators, of which concurrent operators form a subset. The continuous group properties are used in a simple proof that local or noncontextual hidden variables (HVs for short) are incapable of replicating predicted measurement outcomes under covariant transformations. This proof places no restrictions on dimension or particle number except for the general requirement  $N \geq 3$  for GHZ states [37]. This raises the question of whether specific, experimentally accessible GHZ contradictions can be found using discrete subgroups for the same cases: Are the existing gaps fundamental or technical in origin? To answer this question, we present a succession of three methods of proof that together succeed in filling all the gaps. This succession reveals three distinct regimes of

specific contradictions, each more demanding than its predecessor in terms of their requirements for both the number of concurrent  $N$ -body operators, and the number of one-qudit measurement bases. We attempt to minimize both numbers in order to provide the simplest protocols for experimental tests at each level.

In the next section we describe the characteristic rotational symmetry of GHZ states, the covariance of operators, and the identification of concurrent operators. We conclude the section with a formal proof that hidden variables cannot replicate the covariance of operators. In Sec. 3 we derive the three regimes of specific GHZ contradictions, completing the catalog of all  $N \geq 3$  for every  $d$ . We conclude this section with a physical interpretation of the contradictions based on the covariance. In the concluding Sec. 4, we summarize the results and discuss remaining open questions.

## II. ROTATIONAL SYMMETRY

A generalized GHZ state of  $N$  qudits can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |nn\dots n\rangle, \quad (1)$$

where the indices refer to the eigenvalues of the one-qudit Pauli  $Z$  operators,

$$Z = \sum_{n=0}^{d-1} |n\rangle \omega^n \langle n|, \quad \text{where} \quad \omega \equiv \exp(2\pi i/d). \quad (2)$$

Imagining the qudits as  $d$ -component spinors,  $Z$  is proportional to an exponential of the spin operator,  $S_z$ , whose eigenvalues are  $m = S, S-1, \dots, -S$ , with  $2S+1 = d$ . The spectra of  $Z$  and  $S_z$  are related by  $n = S - m$ , so the operators are related by  $Z = \omega^{(SI-S_z)}$ , and rotations about  $\hat{z}$  axes may be written as powers of  $Z$ ,

$$R(\phi) = \exp(-iS_z\phi) = e^{-iS\phi} Z^{\phi d/2\pi}. \quad (3)$$

Rotations of all  $N$  qudits through independent angles  $\phi_k$  about their respective  $\hat{z}_k$  axes ( $k = 1, \dots, N$ ), are products of the individual rotations,  $\mathbf{R}(\{\phi_k\}) = \prod_k R_k(\phi_k)$ . These rotations form the group of  $N$  independent uniaxial rotations,  $\mathcal{T}^{\otimes N}$ , where  $\mathcal{T}$  is the group of rotations in a plane, or the circle group. Applied to the state  $|\Psi\rangle$  above, we find

$$\mathbf{R}(\{\phi_k\})|\Psi\rangle = \frac{1}{\sqrt{d}} e^{-iS\Phi} \sum_{n=0}^{d-1} e^{in\Phi} |nn\dots n\rangle \equiv |\Psi(\Phi)\rangle, \quad \Phi = \sum_{k=1}^N \phi_k, \quad (4)$$

so that the transformed state depends only on the compound (net) rotation angle  $\Phi$ , and reveals nothing about the individual  $\phi_k$ . Because of this, we can represent  $|\Psi(\Phi)\rangle$  uniquely on the unit circle. Moving around from 0 to  $2\pi$ , one encounters  $d$  orthogonal states at integral multiples of  $2\pi/d$ , as illustrated for the case of  $d = 3$  in Fig. 1(a). To show this orthogonality, note that the inner product of any two states on the unit circle is

$$\langle\Psi(\Phi')|\Psi(\Phi)\rangle = \langle\Psi|R(\Phi-\Phi')|\Psi\rangle = \frac{\sin d(\Phi-\Phi')/2}{d\sin(\Phi-\Phi')/2}. \quad (5)$$

This vanishes if and only if the two angles differ by a nonzero multiple of  $2\pi/d$  and confirms that the states  $|\Psi(2\pi\nu/d)\rangle$ , for  $\nu = 0, 1, \dots, d-1$ , form an orthonormal set. Thus,  $|\Psi\rangle$  transforms as a  $d$ -dimensional representation of  $\mathcal{T}$ . Note in passing that Eq. 4 signals a sign change under any compound  $2\pi$  rotation for systems composed of half-integral spin (even- $d$ ) particles.

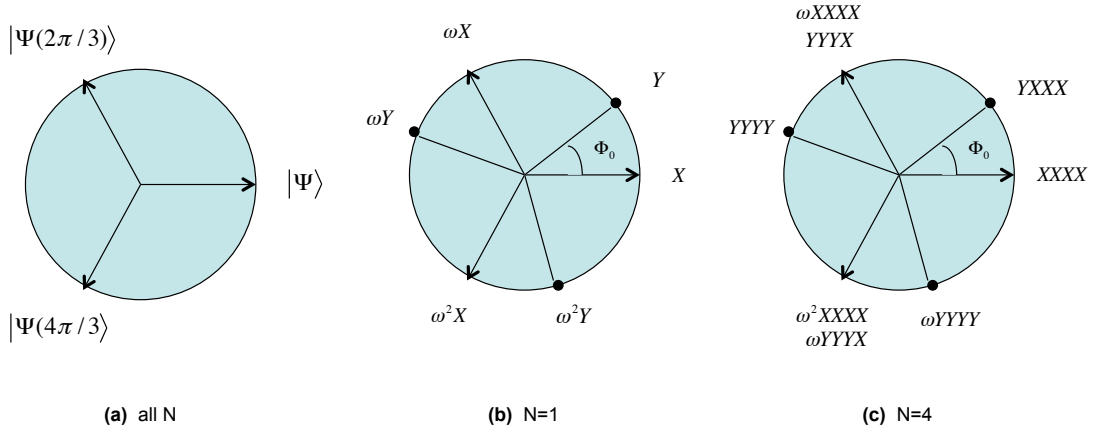


FIG. 1: Circle plots for  $N$  qutrits ( $d = 3$ ): (a) orthonormal set of GHZ states for any  $N$ , (b) periodicity property of operators for one qutrit, and (c) periodicity property of tensor product operators for 4 qutrits. One locates tensor product operators on the plot by adding up the angular variables of all the factors.

Now consider the rotational properties of operators. The crucial operator of which  $|\Psi\rangle$  is an eigenstate (with eigenvalue unity) is the tensor product,

$$\mathbf{X} = \prod_k X_k, \quad (6)$$

where the one-qutrit operators  $X_k$  are the usual raising operators of  $Z_k$ ,

$$X_k = \sum_{n=0}^{d-1} |n+1\rangle\langle n|, \quad (7)$$

with the convention  $|d\rangle \equiv |0\rangle$  understood. We shall refer to  $\mathbf{X}$  and the  $X_k$  as observables because (like  $Z_k$ ) they are unitary, and therefore exponentials of Hermitian operators whose eigenvalues are  $0, 1, \dots, d-1$ . We define covariant rotations of  $\mathbf{X}$  as those which preserve eigenvalue relations in rotated states,

$$\mathbf{R}(\{\phi_k\})\mathbf{X}\mathbf{R}^{-1}(\{\phi_k\}) \equiv \mathbf{X}(\{\phi_k\}) = \prod_k X_k(\phi_k), \quad (8)$$

so that  $|\Psi(\Phi)\rangle$  is an eigenstate of any  $\mathbf{X}(\{\phi_k\})$  for which  $\sum \phi_k = \Phi$ . More generally, considering relative rotations of operator and state, the expectation value  $\langle \Psi | \mathbf{R}(\{\phi_k\})\mathbf{X}\mathbf{R}^{-1}(\{\phi_k\}) | \Psi \rangle$  reflects the fact that rotating *only* the operator has the same effect on measurement outcomes as rotating only the state in the opposite sense.

The rotated one-qudit factors in Eq. 8 are given by

$$X_k(\phi_k) = e^{i\phi_k} \sum_{n=0}^{d-2} |n+1\rangle\langle n| + e^{i(1-d)\phi_k} |0\rangle\langle d-1|. \quad (9)$$

Each factor  $X_k(\phi_k)$  has two key properties: (i) It is  $e^{i\phi_k}$  times a periodic function of  $\phi_k$  with period  $2\pi/d$ :

$$X_k(\phi_k + 2\pi/d) = e^{2\pi i/d} X_k(\phi_k) \equiv \omega X_k(\phi_k), \quad (10)$$

and (ii) it lives in a 2-dimensional operator space spanned by  $X_k$  and another operator,  $Y_k$ , which may be defined at an arbitrary point within the first period,

$$Y_k \equiv X_k(\phi_o), \quad 0 < \phi_o < 2\pi/d. \quad (11)$$

The periodicity is illustrated in Fig. 1(b) for the case of  $d=3$ , where the underlying period is  $2\pi/3$ , and  $Y$  is chosen to be  $X(2\pi/9)$  for the purpose of Sec. III. For  $d > 2$ ,  $X$  and  $Y$  form a nonorthogonal operator basis (whatever the choice of  $\phi_o$ ). For  $d=2$  (not shown), the choice  $Y = X(\pi/2)$  is the usual Pauli matrix, producing an orthogonal basis,  $[\text{Tr}(XY) = 0]$ .

There are infinitely many rotated  $N$ -qudit operators  $\mathbf{X}(\{\phi_k\})$  that correspond to a particular collective angle  $\Phi$ . It is useful to denote the set of all such operators as  $[\mathbf{X}(\Phi)]$ :

$$[\mathbf{X}(\Phi)] \equiv \left\{ \mathbf{X}(\{\phi_k\}) : \sum_{k=0}^{d-1} \phi_k = \Phi \right\}, \quad (12)$$

and to associate this set with the point  $\Phi$  on the unit circle. Examples of discrete subsets consisting of factors of just  $X$ s and  $Y$ s are given in Fig. 1(c). Now, because an element

of the set  $[\mathbf{X}(\Phi)]$  is generated by rotating a single qudit through  $\Phi$ , this set must have the same periodicity property (Eq. 10) as an individual  $X_k$ , that is,  $[\mathbf{X}(\Phi + 2\pi/d)] = \omega[\mathbf{X}(\Phi)]$ . Therefore, at the special points  $\Phi = 2\pi\nu/d$  ( $\nu = 0, 1, \dots, d-1$ ) on the unit circle, we have

$$[\mathbf{X}(2\pi\nu/d)] = \omega^\nu[\mathbf{X}]. \quad (13)$$

Recalling that  $|\Psi\rangle$  (Eq. 1) is an eigenstate of  $\mathbf{X}$  with eigenvalue unity, Eq. 13 shows that it is also an eigenstate of every operator in the set  $[\mathbf{X}(2\pi\nu/d)]$  with eigenvalue  $\omega^\nu$ ,

$$[\mathbf{X}(2\pi\nu/d)]|\Psi\rangle = \omega^\nu|\Psi\rangle. \quad (14)$$

Covariance implies that the GHZ state at *any* special point,  $|\Psi(2\pi\mu/d)\rangle$ , is an eigenstate of operators at  $\nu$  with eigenvalue  $\omega^{\nu-\mu}$ . In terms of concurrency, one could say that the set of operators  $\cup_{\nu=0}^{d-1}[\mathbf{X}(2\pi\nu/d)]$  is concurrent with respect to the set of GHZ states  $\{|\Psi(2\pi\mu/d)\rangle : \mu = 0, 1, \dots, d-1\}$ . For the special case of  $d = 2$  with the choice of Pauli operators, the corresponding set is also compatible.

Equation 14 forms the basis of specific GHZ contradictions, and readers who so desire can skip to Sec. III where that discussion begins, or continue with the formal proof immediately following, which may be useful for later developments.

### General Failure of Hidden Variables

The assumption embodying local realism, or more generally noncontextuality, is that each one-qudit factor takes a definite value,  $v(X_k(\phi))$ . Since this value must be an eigenvalue of  $X_k(\phi)$ , it is natural to parameterize it as

$$v(X_k(\phi)) = \omega^{\mathcal{X}_k(\phi)}, \quad (15)$$

where  $\mathcal{X}_k(\phi) = 0, 1, \dots, d-1$ . To conform to  $N$ -body measurements as described by quantum theory, these variables must be individually random, but correlated so as to reproduce definite  $N$ -body products when the GHZ state under consideration is an eigenstate. Here we shall demonstrate, in general terms, that such variables are incapable of replicating the covariance described above. We shall proceed by demanding that they respect the invariance of measurement outcomes (eigenvalues) under any  $\Phi$ -preserving rotations (we will call this GHZ *invariance*), and then show that they fail to transform covariantly under any  $\Phi$ -changing rotation.



To impose GHZ invariance, we shall consider measurements on the state  $|\Psi\rangle$  corresponding to several operators at  $\nu = 0$ , all of which produce the value 1 with certainty. We choose  $\mathbf{X}$  and any operator obtained from it by rotating two qudits through opposite angles, namely  $\mathcal{O}_{ij} = [X_i(\phi)X_i^{-1}X_j(-\phi)X_j^{-1}]\mathbf{X}$ . Equating the values,  $v(\mathcal{O}_{ij}) = v(\mathbf{X})$ , relates the ratios of individual factors, *eg*,  $v(X_i(\phi))/v(X_i)$ , which depend only on the *variations* of the exponents with angle,  $\Delta\mathcal{X}_i(\phi) \equiv \mathcal{X}_i(\phi) - \mathcal{X}_i(0)$ . The resulting equation is

$$\Delta\mathcal{X}_i(\phi) + \Delta\mathcal{X}_j(-\phi) = 0. \quad (16)$$

Since this equation applies to all  $i$  and  $j$ , the variations are uniform over qudits,

$$\Delta\mathcal{X}_1(\phi) = \dots = \Delta\mathcal{X}_N(\phi) \equiv \Delta\mathcal{X}(\phi), \quad (17)$$

that is, there is a single variation, and this is an odd function of  $\phi$ ,

$$\Delta\mathcal{X}(\phi) = -\Delta\mathcal{X}(-\phi). \quad (18)$$

The variations are thus constrained even though the functions  $\mathcal{X}_i(\phi)$  themselves are random.

Assume now that we have  $N \geq 3$  qudits, divide them into two unequal groups consisting of  $N_1$  and  $N_2$  qudits each, and let  $\lambda = N_2/N_1 > 1$ . Rotate the first group through  $\phi$ , and the second group through  $(-\phi/\lambda)$ , making  $\Phi = 0$ . Using Eqs. 17 and 18, we arrive at

$$\Delta\mathcal{X}(\phi) = \lambda\Delta\mathcal{X}(\phi/\lambda). \quad (19)$$

This equation requires that  $\Delta\mathcal{X}(\phi)$  be a linear function of  $\phi$ . But since it can only take discrete values, it must be a constant, and by definition that constant must be zero. So  $\Delta\mathcal{X}(\phi) = 0$  for every qudit. This means that while every  $\mathcal{X}_k(\phi)$  is a random variable, it must be isotropic. Together, these  $\mathcal{X}_k(\phi)$  predict that the  $N$ -body values  $v(\mathbf{X}(\{\phi_k\}))$  cannot vary with  $\Phi$ , and, in the state  $|\Psi\rangle$ , all are unity. If instead we chose the state  $|\Psi(2\pi\nu/d)\rangle$  for this proof, the predicted value would be  $\omega^\nu$ . In short, hidden variables that respect GHZ invariance must fail the covariance.

It is interesting that this general failure is proven independently of  $d$  and  $N$ , except for the condition that  $N \geq 3$ . Unfortunately, this result does not lead immediately to GHZ contradictions that are congenial to experimental tests. While one can test the invariance of eigenvalues of  $\mathcal{O}_{ij}$  at particular choices of  $\phi$ , thus establishing that  $\mathcal{X}_k(\phi)$  is an element of reality for those choices, this does not prove it for every  $\phi$ . Testable contradictions may be

found in this approach by identifying an appropriate angular interval ( $\phi_o$ ) that defines a finite set of individual factors ( $X$  and  $Y$  in the simplest case), and constructing a finite (minimal) set of  $N$ -body tensor products for the proof. An experimental test requires measurements of all these, and each in turn requires individual qudit measurements in bases dictated by the factors. Since the number of such factors must be kept finite, less can be inferred about individual hidden variable properties than in the proof above. In fact, in existing proofs such inferences need not be made explicitly, although they are present implicitly. In the succession of proofs to follow, we shall see that as more operators are required, more inferences become possible, and that at some level the proofs may be simplified by making these inferences explicit.

### III. GHZ CONTRADICTIONS

One can identify three regimes of GHZ contradictions according to increasing numbers of measurements required for experimental tests. In this section we present three methods of proof that define these regimes. The first proof is similar to that of Ryu et. al. [33]; the contradictions are equivalent and the values of  $N$  and  $d$  are the same. While it is possible to infer HV properties from the concurrent operators used, the proof is simpler without this.

#### Method 1

We choose several tensor products of  $X$ s and  $Y$ s at  $\nu = 1$  (as in Fig. 1c) and obtain the contradiction by comparing with  $\mathbf{X}$  (at  $\nu = 0$ ). To appropriately define the  $Y$  factors for the tensor products, let  $f$  be any factor of  $d$  ( $f \neq 1$ ), and choose  $\phi_o = 2\pi/fd$ , that is,

$$Y_k \equiv X_k(2\pi/fd), \quad (20)$$

which divides the basic period into  $f$  parts. (Fig. 1c corresponds to  $f = d = 3$ , and  $N = 4$ .) Then, for any  $N > f$ , we find many operators at  $\nu = 1$  that have  $f$  factors of  $Y$  and  $N - f$  factors of  $X$ . We need  $N$  such operators, and we select those which have the  $X$  and  $Y$  factors grouped together, as in  $XXXXYY$ ,  $YXXXXY$ , etc., regarding  $Y_1$  and  $Y_N$  as neighbors.

We now introduce hidden variables for the  $X$  and  $Y$  factors,

$$v(X_k) = \omega^{\mathcal{X}_k} \quad \text{and} \quad v(Y_k) = \omega^{\mathcal{Y}_k}, \quad (21)$$

with  $\mathcal{X}_k$  and  $\mathcal{Y}_k = 1, 2, \dots, d-1$ , and compare the requirements at  $\nu = 0$  and 1: Assuming that the system is in state  $|\Phi\rangle$ , in which  $\mathbf{X}$  has eigenvalue unity, we have

$$\sum_k \mathcal{X}_k = 0. \quad (22)$$

The other  $N$  operators all have eigenvalue  $\omega$  in  $|\Psi\rangle$ . Rewriting these eigenvalue equations in terms of  $\mathcal{X}_k$  and  $\mathcal{Y}_k$  and then adding them all together, we find

$$(N-f) \sum_k \mathcal{X}_k + f \sum_k \mathcal{Y}_k = N. \quad (23)$$

Combining 22 and 23 leaves us with

$$f \sum_k \mathcal{Y}_k = N, \quad (24)$$

with equality understood modulo  $d$ . This equation has solutions only if  $N$  is a multiple of  $f$ . Therefore, we have GHZ contradictions for all  $N > f$  that are *not* multiples of  $f$ .

If  $d$  is a prime, then the above applies simply to  $f = d$ , and contradictions are found as shown in the prime- $d$  columns of Fig. 2, with  $N$ -values denoted by red squares.

If  $d$  has multiple factors, then the above proof applies to all of them, using different  $\phi_o$  values for each. Any one of these options suffices to establish a contradiction. Therefore, contradictions begin with  $N = p_1 + 1$ , where  $p_1$  is the smallest (prime) factor of  $d$ , and above this they occur for every  $N$  that is not a multiple of *every* factor of  $d$  below  $N$ . These contradictions are also represented by the red squares in Fig. 2. The remarkable case of  $d = 12$  is understood in terms of the factors 2, 3, and 4.

One finds many other specific GHZ contradictions with this method, but I have found none with further  $N$  values for any  $d$ . To illustrate, consider examples with  $d = 3$ : When  $N = 4$ , Fig. 1(c) shows a trivial alternative to the standard construction using operators at  $40^\circ$  and  $160^\circ$ . When  $N = 5$ , a similar trivial alternative occurs at  $80^\circ$  and  $200^\circ$ ; less trivially, at  $40^\circ$  and  $160^\circ$ , one finds operators of the type  $YXXXX$  and  $YYYYX$  respectively. These provide five HV equations at each point, clearly different from Eqs. 22 and 23, but their combination reduces to the same final condition 24. In the more interesting case of  $N = 6$ , although we have the choice of five distinct sets of concurrent operators, the corresponding sets of equations all reduce to the standard one, which shows *no* GHZ contradictions. Method 2 is complementary in providing proofs for just such cases.

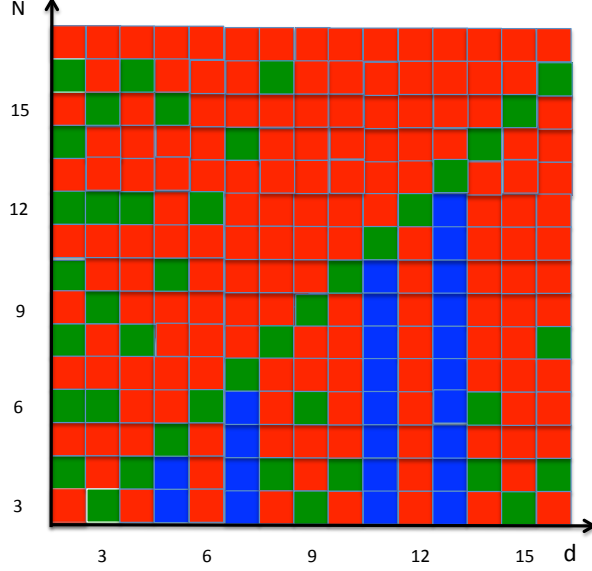


FIG. 2: Red indicates known contradictions (Refs. [27] and [33]) recovered by method 1. Green and blue indicate contradictions found by methods 2 and 3, respectively.

### Method 2: Multiples of $d$ and its factors

Here we derive GHZ contradictions, again for any  $d$ , where  $N$  is a multiple of any factor of  $d$ . Regime 2 (the green squares) is defined as the set of all such cases not already assigned to regime 1 (and colored red). Using different methods, Refs. [29, 30, 34] have found contradictions for subsets of regime 2 [38]. In order to fill remaining gaps and provide a single derivation covering all of regime 2, we introduce a new type of derivation which, in the spirit of the formal proof of Sec. 2, makes explicit reference to hidden variable failure.

In analogy with the formal proof, we define a conjugate of  $Y$ , namely  $\tilde{Y} \equiv X(-\phi_o)$ , and use it to form a set of concurrent operators at  $\nu = 0$ , starting with

$$O_1 = YX...X\tilde{Y}, O_2 = XYX...X\tilde{Y}, \dots, O_{N-1} = X...XY\tilde{Y}, \quad (25)$$

assuming that  $N \geq 3$ . Equalities among the values  $v(\mathcal{O}_k)$  and  $v(\mathbf{X})$  relate the ratios of individual qudit factors, which in turn depend only on variations in the exponents, defined here by

$$\ln_\omega[v(Y_k)/v(X_k)] = \mathcal{Y}_k - \mathcal{X}_k \equiv \Delta_k, \quad (26)$$

$$\ln_\omega[v(\tilde{Y}_k)/v(X_k)] = \tilde{\mathcal{Y}}_k - \mathcal{X}_k \equiv \tilde{\Delta}_k. \quad (27)$$

From these equalities, we may deduce that

$$\Delta_1 = \Delta_2 = \dots = \Delta_{N-1} = -\tilde{\Delta}_N. \quad (28)$$

The equalities can be extended to include  $\Delta_N$  by adding two more operators,

$$O_N = X \dots X \tilde{Y} Y \quad \text{and} \quad O_{N+1} = Y X \dots X \tilde{Y} X, \quad (29)$$

from whose ratios we deduce that all of the  $\Delta$ s are equal,

$$\Delta_k = \mathcal{Y}_k - \mathcal{X}_k \equiv \Delta, \quad (30)$$

that is, the HV variations are uniform over qudits. A similar deduction for the values of individual  $\mathcal{X}_k$  and  $\mathcal{Y}_k$  is of course impossible, since these must be random.

To find GHZ contradictions, we may choose  $\phi_o = 2\pi/Nd$ , so that  $Y^{\otimes N}$  appears at  $\nu = 1$  and the quantum prediction for its measured value is  $Y^{\otimes N} \rightarrow \omega$ . The hidden variables prediction based on Eq. 30 is

$$v(Y^{\otimes N}) = v(X^{\otimes N}) \omega^{\sum_k \Delta_k} = \omega^{N\Delta}. \quad (31)$$

Consistency with the quantum prediction requires that

$$N\Delta = 1, \quad (32)$$

which cannot be satisfied for  $N$  equal to any multiple of  $d$ , or of its factors (excluding unity). Note that  $N = 2$ , which admits a hidden variable construction of perfect correlations [37], is ruled out by construction (Eqs. 25 and 29). We have thus derived GHZ contradictions for those values of  $N$  which elude method 1, down to the smallest factors of  $d$ , as shown in green in Fig. 2.

The derivation described above requires  $N + 3$  observables, composed of three measurement bases for two of the qudits, and two measurement bases for remaining qudits. In comparison, method 1 required only  $N + 1$  observables, with two measurement bases for every qudit. These requirements may be taken as the operational definitions of regimes 1 and 2. Cases for which both methods work are assigned to regime 1.

### Method 3: All $N \geq 3$

Here we address the remaining cases colored blue in Fig. 2. These require still further concurrent operators, and correspondingly further one-qudit measurement bases, whose numbers will be estimated later.

Let us illustrate this method for the most challenging case,  $N = 3$ , from which it will be clear how to generalize. Concurrent operators at  $\nu = 0$  from the previous section are

$$YX\tilde{Y}, XY\tilde{Y}, X\tilde{Y}Y, Y\tilde{Y}X. \quad (33)$$

Measured values of unity allow the inference that  $\Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta$ . The necessary additional concurrent operators are built with one-qudit factors defined at multiples of the basic angle  $\phi_o$ . We write these as  $Y^{(n)} \equiv X(n\phi_o)$ , for  $n = \pm 1, \pm 2, \dots$ , and then define the Fibonacci-like sequence of operators, all at  $\nu = 0$ ,

$$Y^{(\pm 2)}Y^{(\mp 1)}Y^{(\mp 1)}, Y^{(\mp 2)}Y^{(\pm 3)}Y^{(\mp 1)}, Y^{(\mp 2)}Y^{(\mp 3)}Y^{(\pm 5)}, Y^{(\pm 8)}Y^{(\mp 3)}Y^{(\mp 5)}, \dots, \quad (34)$$

from which the necessary operators can be chosen. Writing the usual hidden variable parameters for the new one-qudit factors as

$$\ln_\omega[v(Y_k^{(n)})/v(X_k)] = \mathcal{Y}_k^{(n)} - \mathcal{X}_k \equiv \Delta_k^{(n)}, \quad (35)$$

we can make the following inferences from measured values of unity on operators in (34): From the first pair,  $\Delta_1^{(2)} = 2\Delta = -\Delta_1^{(-2)}$ ; from the second pair,  $\Delta_2^{(3)} = 3\Delta = -\Delta_2^{(-3)}$ ; and so on. We can stop as soon as the indices add up to  $d$ : If  $d = 5$ , then just two elements from the sequence,  $Y^{(-2)}YY$  and  $Y^{(-2)}Y^{(3)}Y^{(-1)}$ , suffice to build the hidden variable prediction for  $YY^{(3)}Y$ , which is

$$v(YY^{(3)}Y) = v(\mathbf{X})\omega^{5\Delta} = 1. \quad (36)$$

For the quantum prediction, we can choose  $\phi_o = 2\pi/25$ , which places  $YY^{(3)}Y$  at  $\nu = 1$ , with eigenvalue  $\omega$ , and forms a GHZ contradiction. We have obtained this contradiction with a total of 8 concurrent operators. In the case of  $d = 7$ , one requires four elements from the sequence (those containing  $Y_1^{(\pm 2)}$ ,  $Y_2^{(-3)}$ , and  $Y_3^{(5)}$ ) to build  $YYY^{(5)}$  for comparison with  $\mathbf{X}$ . The hidden variable and quantum predictions, with  $\phi_o = 2\pi/49$  for the latter, form a similar GHZ contradiction with a total of 10 concurrent operators. For larger  $d$ , the choice  $\phi_o = 2\pi/d^2$  remains appropriate.

Contradictions with larger values of  $N$  are straightforward extensions: We employ  $N + 2$  concurrent operators of method 2 (Eqs. 25 and 29 and  $\mathbf{X}$ ), and select additional operators from a similar Fibonacci-like sequence whose number will increase with  $d$  but decrease with  $N$ . The largest  $N$  values for which this method is useful are  $N = d - 1$ . In these cases, only a single operator is required from the sequence, for example  $Y^{(2)}X\dots X\tilde{Y}\tilde{Y}$ , allowing a hidden

variable prediction of the  $\nu = 1$  operator,  $Y^{(2)}Y\dots Y$ , to demonstrate the contradiction with a total of only  $N + 4$  concurrent operators. The choice of  $\phi_o = 2\pi/d^2$  remains appropriate for all relevant  $N$  ( $3 \leq N < d$ ).

Note that the above inferences  $\Delta_k^{(n)} = n\Delta$  involving multiples of  $\phi_o$  are particle *specific*, a consequence of minimizing the number of required measurements. In principle these inferences could be extended to all qudits by including further operators in the sequence, implying uniformity of variations as well as linearity as a function of the discrete angle,  $n\phi_o$ . This goes beyond what we need for the desired GHZ contradictions, but, of course, falls short of the proof  $\Delta\mathcal{X}_k(\phi) = 0$  based on a continuum of angles.

### Compared Requirements

Let us compare the three regimes with respect to the minimal requirements for each method of proof described above. For each method (1, 2, 3), the minimum number of concurrent  $N$ -body operators is  $N + 1$ ,  $N + 3$ , and  $\geq N + 4$ , respectively. The corresponding number of one-qudit measurement bases is  $2^N$ ,  $2^{N-2}3^2$ , and  $(\geq 2)^{N-3}(\geq 3)^3$ , where superscripts denote the number of particles to which the basis number applies. Specifically, method two requires three bases for two qudits and two bases for all others, while method three requires at least three bases for three qudits and at least two for all others. In method 3, as the examples show, the number of operators and measurement bases depend on both  $d$  and  $N$ , generally as increasing functions of  $d$  and decreasing functions of  $N$ .

Note that as we progress through the three methods, the possible inferences are expanded, whether or not one chooses to make use of them in the proofs. One can easily see that the concurrent operators employed in method 1 allow the inference that some or all of the variations  $\Delta_k$  are equal to one another [39]. With method 2, we infer the uniformity of both  $\Delta_k$  and  $\tilde{\Delta}_k$ . With method 3 we infer in addition the linearity ( $\Delta_k^{(n)} = n\Delta_k$ ) for some, but not necessarily all of the qudits  $k$ .

### Irreducibility of Contradictions

An  $N$ -particle contradiction is irreducible if no single qudit can be removed without spoiling it. Those of method 1 were shown to be irreducible in Ref. [33]. Those of method 2 are clearly irreducible because every qudit has both  $X$  and  $Y$  factors from at least one of the concurrent operators, all of which are essential to the proof. As for method 3, it is

difficult to argue in general because of its inherent flexibility. But it is easy to see that all of the given examples are irreducible. It seems plausible that if one minimizes the number of operators used in the proof, it will also be irreducible with respect to  $N$ .

A contradiction is genuinely  $d$ -dimensional if the  $d \times d$  matrices  $X_k$  and  $Y_k$  cannot be simultaneously block-diagonalized, or equivalently, if one cannot find an eigenstate of  $X_k$ , and another of  $Y_k$ , with vanishing inner product [33]. Fig. 1 (a and b) provides an elegant general proof that indeed one cannot: Note that Eq. 4 applies to the  $N = 1$  case, where the rotated states (a) are eigenstates of the rotated  $X_k(\phi_k)$  matrix (b). Covariance shows that the states at the special points  $2\pi\nu/d$  form a basis of eigenstates of  $X_k$ , while those at points  $\phi_o + 2\pi\nu/d$  form a basis for  $Y_k$ . Equation 5 shows that the inner product of any pair of states, one from each basis, does not vanish. Since both bases are nondegenerate, there is no other choice of eigenstates. So our contradictions are genuinely  $d$ -dimensional. This argument applies equally well to other one-qudit factors  $Y^{(n)}$ .

### Physical Interpretation

As we have seen, the requirement for GHZ contradictions is the multiplicity of  $N$ -body operators that share a common eigenstate, with two or more differing eigenvalues. To relate this to rotational covariance, note that if we rotate  $|\Psi\rangle$  through an angle  $(-2\pi/d)$ , it is still an eigenstate of  $\mathbf{X}$ , but with eigenvalue  $\omega$  rather than 1. Equivalently, if we rotate  $\mathbf{X}$  through  $(+2\pi/d)$ , it still has  $|\Psi\rangle$  as an eigenstate, but with eigenvalue  $\omega$ . The difference is that in the latter case, there many operators arising from the many ways of distributing the net rotation among the factors. All of these rotated operators correspond to the same equivalently rotated state, which is oblivious to the distribution. Thus, the multiplicity arises from the invariance of the GHZ state under the  $\Phi$ -preserving rotations that relate all of the operators at  $(+2\pi/d)$ .

The GHZ contradictions require in addition that operators at different points (separated by  $2\pi/d$  or a multiple) have different eigenvalues in the same state, a property of GHZ covariance. The common feature of all successful proofs is that HV functions  $[\mathcal{X}_k(\phi)]$  are so constrained by the quantum predictions at any one point, that they cannot reproduce those at another point.

For a broader perspective, let us return to the discussion of continuous transformation properties of Sec. II, where we showed that states transform according to the circle



group,  $\mathcal{T}$ , while operators transform as  $\mathcal{T}^{\otimes N}$ . We show here that the expectation value,  $\langle \Psi | \mathbf{R}(\{\phi_k\}) \mathbf{X} \mathbf{R}^{-1}(\{\phi_k\}) | \Psi \rangle$ , transforms simply as  $\mathcal{T}$  under *relative* rotations ( $\Phi$ ) between the operator and the state. Note first that the one-body operator, Eq. 9, and hence its expectation value in the fixed one-particle state, Eq. 1, transform as two-dimensional representations of  $\mathcal{T}$ . Second, the arguments leading to Eq. 13 show that the expectation values of  $N$ -body operators  $[\mathbf{X}(\Phi)]$  in fixed  $N$ -particle GHZ states (1) are given by the same function of  $\Phi$ , independent of  $N$ . This result generalizes to the probability distribution of measurement outcomes for rotated  $N$ -body operators in fixed GHZ states, whether these outcomes are definite or probabilistic. In this respect, the  $N$ -body outcomes are reduced to one-body outcomes.

#### IV. CONCLUSIONS

In summary, we have shown that the many-particle rotational symmetry characterizing GHZ states cannot be satisfied by hidden variables for  $N \geq 3$  particles of any dimension  $d$ . A discrete subset of symmetry operations involving net rotations of  $2\pi/d$  identifies concurrent operator sets that exhibit specific, experimentally verifiable GHZ contradictions that, in total, are similarly unrestricted. These contradictions fall into three regimes defined according to increasing numbers of  $N$ -body operators as well as one-qudit measurement bases required for theoretical proofs and experimental tests. The first regime recovers existing proofs [27, 33], the second regime adds new proofs to some existing ones [38], and the third regime consists entirely of new proofs that complete the catalog of all possible  $N$  values for every  $d$ .

The current results are interesting in part because of the novelty of the odd- $d$  contradictions (of Ref. [33] as well as the present work), which demonstrate that the concurrent operator approach places almost all dimensions on equal footing with respect to the existence of GHZ contradictions. State-dependent contradictions exist for all  $d$ , and state-independent for none, with the exception of  $d = 2$ , where Pauli operators are recovered by appropriate rotations. In contrast, there appears to be a fundamental distinction between even and odd dimensions when applying stabilizer sets in higher dimensions, in that state-independent contradictions have been found for all even  $d$  [26, 29, 32], but shown not to exist for any odd  $d$  [36]. A limited number of state-dependent contradictions have been found for some

odd  $d$  [35].

The successful use of GHZ symmetry in the present work raises the question of a more general relationship between entangled-state symmetries and GHZ contradictions. Will such particular symmetries more generally favor concurrent operator sets over stabilizer sets? If so, then since entanglement and nonlocality are useful resources for quantum information processing, perhaps concurrent operators will prove useful as well.

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- [38] Referring to the green squares in Fig. 2, Ref. [29] proved the even- $d$  cases, and Ref. [34] proved odd- $N$  cases below the line  $N = d + 1$ . These and remaining cases are covered by the alternative method described in the text.
- [39] If  $f$  and  $N$  are coprime, then one may infer that all  $\Delta_k$  are equal. If not, then uniformity can still be inferred within subsets.