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A Notion of Rectifiability Modeled on Carnot Groups

Scott D. Pauls
Dartmouth College

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SCOTT D. PAULS

Abstract. We introduce a notion of rectifiability modeled on Carnot groups. Precisely, we say that a subset E of a Carnot group G and N is a subgroup of a Carnot group N', we say E is N-rectifiable if it is the Lipschitz image of a positive measure subset of N. We prove two main theorems. First, the property that $E \subset G$ is M-rectifiable, where M is a Carnot group (not merely a subgroup of a Carnot group), is equivalent to M-approximability and the existence of approximate tangent cones isometric to M almost everywhere in E. Second, we investigate the rectifiability properties of level sets of Lipschitz functions, $f : G \to \mathbb{R}$, where G is the Heisenberg group of dimension $2n+1$. We show that for almost every $t \in \mathbb{R}$ and almost every $x \in f^{-1}(t)$, there exists a subgroup H of G and $r > 0$ so that $f^{-1}(t) \cap B_G(x,r)$ is H-approximable and has approximate tangent cones isomorphic to H almost everywhere.

1. Introduction

The notion of rectifiability is central to the study of standard geometric measure theory, allowing for the proof of classical geometric properties in a much more general setting. In recent years, there has been significant interest and progress in the study of rectifiable sets not only in Euclidean space but in more general metric spaces as well (see, for example, [Amb], [AK99b], [AK99a], [Che99], [DS97], [FSSC99], [GN96], [Kir94], [Mag00], [Pan89], [Whi98]. Also, see the extensive bibliographies in [GN96] and [AK99b]). In attempting to use the techniques of geometric measure theory to investigate the properties of general metric spaces, one quickly encounters a major difficulty: there may not be any rectifiable subsets or the set of rectifiable subsets may be too small to reveal any significant geometry. Here, we consider a subset of a metric space to be rectifiable if it can be realized as the Lipschitz image of a piece of Euclidean space. Thus, to have any hope of transporting the techniques of Euclidean geometric measure theory to metric spaces, we need a more general notion of rectifiable sets which are modeled on a wider class of metric spaces than simply Euclidean ones. In this paper, we investigate a special situation where much of the standard rectifiable theory carries over, but reveals some of the complications inherent in this endeavor.

We will restrict ourselves to the investigation of the so-called Carnot groups - connected, simply connected, graded nilpotent Lie groups equipped with a left-invariant Carnot-Carathéodory metric (see below for precise definitions). Carnot groups arise in a variety of situations: in the asymptotic geometry of manifolds of negative curvature, in optimal control theory, in the local geometry of equiregular Carnot-Carathéodory manifolds, in

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CR geometry, in the study of hypoelliptic partial differential equations, and in many other areas. In addition, Carnot groups are analytically very similar to Euclidean spaces (which are themselves particular examples of Carnot groups) in that they possess translations and dilations that respect the metric. Because of these similarities to Euclidean spaces, Carnot groups form a good test class for the generalization of the notion of rectifiability.

Further motivation for considering rectifiability modeled on Carnot groups arises in the study of the local geometry of Carnot-Carathéodory manifolds. A Carnot-Carathéodory manifold is a quadruple $(M, \mathcal{S}, < \cdot, \cdot >, d_M)$ where M is a smooth manifold, $\mathcal{S}$ is a subbundle of the tangent bundle, $< \cdot, \cdot >$ is a smoothly varying inner product on the fibers of $\mathcal{S}$ and d_M is the path metric formed by taking the infimum of lengths of paths (calculated with respect to $< \cdot, \cdot >$) among paths tangent a.e. to $\mathcal{S}$. Further, letting $d_i(x)$ be the dimension of the subspace of T_xM spanned by all the commutators of order less than or equal to i, we say that $(M, \mathcal{S}, < \cdot, \cdot >, d_M)$ is equiregular if the vector $(d_0(x), d_1(x), ...)$ is locally constant. Carnot groups are very special examples of equiregular Carnot-Carathéodory spaces given by the quadruple $(G, \mathcal{V}, < \cdot, \cdot >, d_G)$ where $\mathcal{V}$ is the bottom level of the grading, thought of as a left invariant subbundle of TG, and $< \cdot, \cdot >$ is a left invariant inner product on $\mathcal{V}$. For general equiregular Carnot-Carathéodory spaces, it is known that the tangent cone to $(M, \mathcal{S}, < \cdot, \cdot >, d_M)$ is isometric to a Carnot group. However, the tangent group may vary from point to point (quite badly!) - see, for example, [Bel96] and [Var81]. Naturally, this makes an investigation of the local geometry of Carnot-Carathéodory manifolds much more difficult than, for example, that of Riemannian manifolds. As we shall see, the framework of rectifiability modeled on Carnot groups identifies a subclass of CC manifolds which possess uniform local behavior.

Let (G, d) be a Carnot group and let N be a subgroup of a Carnot group, (N', d'). We define the following generalization of rectifiability: a subset $S \subset G$ is N-rectifiable if it is the Lipschitz (with respect to d and d') image of a positive measure subset of N. This clearly generalizes the standard notion of rectifiability where G and N are replaced by Euclidean spaces of the appropriate dimension. Using this terminology, we will refer to the standard theory of rectifiability as $\mathbb{R}^k$-rectifiability. The main results of this paper concern the properties of N-rectifiable sets, recovering many of the basic facts about $\mathbb{R}^k$-rectifiable sets in $\mathbb{R}^n$ such as unique approximate tangent cones, approximative qualities of tangent cones, and the rectifiability of level sets of Lipschitz functions. Precisely, we show the following theorems.

Theorem A. *Let N and G be Carnot groups and suppose $E \subset G$ is a subset with nonzero Hausdorff measure. Then, the following are equivalent*

- *E is N-rectifiable*
- *E is N-approximable*
- *For almost every $x \in E$, there exists a unique approximate tangent cone at x which is isomorphic to N.*

Roughly, a subset is N-approximable at a.e. point if there is a copy of N sitting in G which approximates E locally in a measure theoretic sense. See section 4 for a precise definition. Such a notion is used in [Mat95] in the case of $\mathbb{R}^k$-rectifiability. The reader should note that this theorem is restricted to N which are full Carnot groups, not proper subgroups. As mentioned above, this theorem, among other applications, provides a class of

Carnot-Carathéodory spaces which are quite well behaved (with respect to local geometry). The class of N-rectifiable submanifolds in a given Carnot group G has the property that each member is a Carnot-Carathéodory manifold in its own right (by restriction of the distribution) and that the tangent cone at almost every point is isomorphic to N.

Theorem B. *Let G be the Heisenberg group of dimension $2n+1$ and let $f : G \to \mathbb{R}$ be a Lipschitz map. Then, for a.e. $t \in \mathbb{R}$ and a.e. $x \in f^{-1}(t)$, there exists a subgroup $T_x \subset G$ and $r_x > 0$ such that $B_N(x, r_x) \cap f^{-1}(t)$ is T_x-approximable and $f^{-1}(t)$ has a unique approximate tangent cone isomorphic to T_x at a.e $x \in B_N(x, r_x) \cap f^{-1}(t)$.*

Note that in this theorem, we must allow N to be a proper subgroup of G and that N, equipped with the metric from G restricted to N, may not be a Carnot group. This theorem is a step towards generalizing the techniques of geometric measure theory to the Carnot setting (in the special case of the Heisenberg groups). In particular, this theorem gives information concerning generalizing the notion of slicing manifolds by Lipschitz maps. Unfortunately, as shown by the limitations in this theorem (and illustrated by an example in section 5) we cannot conclude that the level sets are N'-rectifiable for some N' - we lack a Lipschitz map. However, as evidenced by the theorem, many of the approximative qualities of rectifiable sets are inherited by the level sets. This suggests a modification of the notion of rectifiable currents in the Carnot setting based on these types of properties. This idea will be explored in a later paper.

The proofs of these theorems rest on extensions of Euclidean analytic tools to the Carnot case. The most useful one of these is the (suitably defined) differentiability of Lipschitz maps on Carnot groups which is originally due to Pansu ([Pan89]). The form of the theorem used in this paper is an extension due to Vodopyanov and Ukhlov ([VU96]) and recently proved using a different method by Magnani ([Mag00]). In addition, in sections 2 and 3, we prove various lemmas concerning the properties of the Hausdorff measure including a metric area formula. Most of the proofs of these lemmas are adaptations of arguments in Federer ([Fed69]) and the proof of the area formula follows Kirchheim's argument in [Kir94]. To prove theorem A, we follow arguments based on arguments in [Mat95] extended using the lemmas and techniques described above. However, to prove theorem B, we diverge from the classical arguments, instead using smooth approximations of the Lipschitz map (as in [FSSC95], [FSSC99] and [GN96]) and apply metric arguments akin to those in [Pau98].

The structure of the paper is as follows: section 2 reviews some of the known measure theory and differentiability results for Carnot groups and proves many of the useful measure theoretic lemmas needed in the proofs, such as properties of Jacobians of maps and a weak Sard-like property. Section 3 is devoted to an area formula for Lipschitz maps between Carnot groups. Sections 4-6 introduce N-rectifiability, N-approximability and proves theorem A. Section 7 is devoted to proving theorem B.

2. Background results for Carnot groups

2.1. **Distances and measures.** Assume that N is a connected, simply connected graded nilpotent Lie group. Recall that N is graded if the Lie algebra decomposes as $\mathfrak{n} = \mathcal{V} \oplus \mathcal{V}_2 \oplus \ldots \oplus \mathcal{V}_n$ where $[\mathcal{V}_i, \mathcal{V}_j] \subset \mathcal{V}i + j$. We denote by $\mathcal{V}$ not only the bottom level of the grading, but the left invariant vector bundle generated by left translating $\mathcal{V}$ around N. We also assume that $\mathcal{V}$ is equipped with an inner product $< \cdot, \cdot >$, which we also think of as a left invariant inner product on the subbundle $\mathcal{V}$.

Definition 2.1. *A Carnot group is a quadruple $(N, \mathcal{V}, < \cdot, \cdot >, d_N)$ where N,$\mathcal{V}$ and $< \cdot, \cdot >$ are as above. To define d_N we let $\mathcal{H}$ be the paths which are tangent almost everywhere to $\mathcal{V}$. Then, the Carnot-Carathéodory distance is defined as*

$$d_N(n_1, n_2) = \inf \left\{ \int < \gamma', \gamma' >^{\frac{1}{2}} \middle| \gamma \in \mathcal{H} \text{ and } \gamma \text{ connects } n_1 \text{ to } n_2 \right\}$$

It follows from the construction that d_N is a left invariant metric on N which admits a homothety, denoted h_t. The homothety is defined by its action on the Lie algebra where it acts on vectors in $\mathcal{V}_i$ by multiplication by t^i.

While the definition of d_N is geometrically compelling, it is very difficult to compute with. Luckily, one can use any of a family of quasi-norms on N to aid in computation. We describe one here.

Definition 2.2. *Suppose N is a Carnot group with grading $\mathfrak{n} = \mathcal{V} \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_n$. For $n \in N$, decompose n according to the Euclidean vector space basis for $\mathfrak{n}$, $n = e^{v_1 + v_2 + v_3 + ... + v_m}$. Then, we define a quasi-norm on N as follows:*

$$|n|_{qn} = \left(\sum_{i=1}^{m} ||v_i||^{\frac{2}{d(v_i)}} \right)^{\frac{1}{2}}$$

where $d(v_i)$ is the level of the grading of which v_i is a member and $|| \cdot ||$ is the Euclidean norm. We also define a function on $N \times N$ by $d_{qn}(n_1, n_2) = |n_1^{-1} n_2|_{qn}$.

Note that d_{qn} is, by construction, left invariant and admits h_t as a homothety. Thus, d_N and d_{qn} are biLipschitz equivalent. A good reference on quasi-norms on Lie groups is [Goo76].

Because Carnot-Carathéodory metrics and quasi-norms on Carnot groups are left invariant, the Hausdorff measures associated to them are in fact all constant multiples of the Haar measure on such a group. Thus, many of the same measure theoretic results concerning densities, etc. are true for $\mathscr{H}^k_N$ that are true for Lebesgue measure in $\mathbb{R}^m$. For proofs and discussions of these facts, in a more general setting, see [DS97]. We will use the biLipschitz equivalence of d_N and d_{qn} and the relation between the Hausdorff measures freely in the computations below.

Lemma 2.3. *Let k be the Hausdorff dimension of N. If $U \subset N$ is $\mathscr{H}^k_N$ measurable then almost every point of U is a Lebesgue density point. In other words, for almost every $a \in U$,*

$$\lim_{r \to 0} \frac{\mathscr{H}^k_N(U \cap B_N(a, r))}{\mathscr{H}^k_N(B_N(a, r))} = 1$$

Next, we define the usual densities.

Definition 2.4. *If N is a Carnot group and $U \subset N$, $x \in N$, we define*

$$\Theta^{*,k}_N(U, x) = \overline{\lim}_{r \to 0^+} \frac{\mathscr{H}^k_N(U \cap B_N(x, r))}{r^k}$$

and

$$\Theta^k_{*,N}(U,x) = \underline{\lim}_{r\to 0^+} \frac{\mathscr{H}^k_N(U\cap B_N(x,r))}{r^k}$$

If both of these coincide, then the common value is denoted $\Theta^k_N(U,x)$.

Note that, due to the lack of normalization of the Carnot metrics and the related Hausdorff measures, our estimates will be much less precise than in the Euclidean case.

Lemma 2.5. *If* $U\subset N$ *and* $\mathscr{H}^k_N(U)<\infty$ *then there exists a constant* $C>0$, *depending only on the structure of the group* N *and the Carnot-Carathéodory metric on* N *so that*

1. $C2^{-k}\le \Theta^{*,k}_N(U,x)\le C$ *for almost all* $x\in U$.
2. *If* U *is* $\mathscr{H}^k_N$ *measurable then* $\Theta^{*,k}_N = 0$ *for almost all* $x\in N\setminus U$.

The proof of this theorem follows from the standard arguments in $\mathbb{R}^n$ (see e.g. [Fed69], 2.10.18 or [Sim83], theorem 3.6) combined with the observation that the Hausdorff measure derived from any Carnot-Carathéodory metric on a Carnot group is left invariant and scales with the homothety.

Next we quote two useful formulae. Both can be found in [Hei95] as equation 4.8 and proposition 4.9 respectively.

To state them, we assume N is a Carnot group and pick a Riemannian completion of the inner product defining the Carnot-Carathéodory metric on N. Let ∇f denote the Riemannian gradient of a function f and denote by $\nabla_0 f$ the horizontal component of ∇f. Below $|\cdot|$ is the usual norm on $\mathbb{R}$.

Proposition 2.6 (Coarea formula)**.** *Suppose* $f: N\to\mathbb{R}$ *is a smooth map from a Carnot group of Hausdorff dimension* k *to the reals and* u *is any nonnegative measurable function. Then,*

$$\int_N u(x)|\nabla_0 f(x)|d\mathscr{H}^k_N(x) = \int_0^\infty \int_{f^{-1}(t)} u(y)d\mathscr{H}^{k-1}_N(y)dt$$

Proposition 2.7. *Suppose* S *is a level surface of a smooth real valued function* f *on* N, *a Carnot group of Hausdorff dimension* k. *Then, for* $\mathscr{H}^{k-1}_N$ *almost everywhere on* S,

$$d\mathscr{H}^{k-1}_N = \frac{|\nabla_0 f|}{|\nabla f|}dA$$

where dA *is the Riemannian area element on* S.

2.2. **Differentiability of Lipschitz maps.**

We first quote a definition and a result of P. Pansu (see [Pan89]) crucial to our constructions:

Definition 2.8. *A map* $f: N\to M$ *between Carnot group is said to be differentiable in the sense of Pansu at* $n\in N$ *with differential* df_n *if the limit*

$$df_n(y) = \lim_{s\to 0} h'_{\frac{1}{s}} f(n)^{-1} f(nh_s y)$$

exists and convergence is uniform for all $y\in N$.

The following theorem is a generalization of a theorem of Pansu ([Pan89]) due to Vodopyanov and Ukhlov ([VU96]). Recently, Magnani ([Mag00]) has also proved this theorem using a different technique.

Theorem 1. *If N and M are Carnot groups and f is a Lipschitz map from a measurable set $U \subset N$ to M, then f is differentiable almost everywhere (in the sense of Pansu) and the differential df_x is a graded group homomorphism at almost every point.*

In [Pau98], the author proves a limited metric differentiability for Lipschitz maps of Carnot groups into complete metric spaces. In the special case when the target space is another Carnot group, either an extension of the author's arguments or an appeal to the result above gives complete metric differentiability. In some of the later results, it is more convenient to work with the metric version of differentiability hence we state it here.

Theorem 2. *Let $f : U \subset N \to M$ be a Lipschitz map between Carnot groups. Then f is metrically differentiable almost everywhere. In other words, for almost every $n \in N$,*

$$\Delta_n(y_1, y_2) = \lim_{t \to 0} \left\{ \frac{d_M(f(n'h_t y_1), f(n'h_t y_2))}{t} \middle| n' \in B_N(n,t) \right\}$$

exists, the limit converges uniformly and Δ_n admits a homothety and is left invariant under the action of N.

The following lemma dictates exactly how the image of the differential approximates the function at a point.

Lemma 2.9. *Let $f : U \subset N \to M$ be a Lipschitz mapping between Carnot groups, where U is a positive measure subset of N and let df_x denote the Pansu differential at a point of differentiability, x. Then,*

$$d_M(f(xe^v), f(x)df_x(e^v)) = o(d_N(e^0, e^v))$$

Proof: This is a straightforward computation. Let v_0 be the Lie algebra vector in the direction of v such that $d_N(e^0, e^{v_0}) = 1$ and let $t = d_N(e^0, e^v)$.

$$\begin{aligned} d_M(f(xe^v), f(x)df_x(e^v)) &= d_M(f(x)^{-1}f(xh_te^v), df_x(h_te^{v_0})) \\ &= d_M(h_t df_x(e^{v_0}), df_x(h_te^{v_0})) + o(t) \\ &= o(t) \end{aligned}$$

The last equality stems from the fact that Pansu's differential intertwines the homotheties of N and M. □

Remark: One should note that the resulting metric Δ_x is very close to being a Carnot-Carathéodory metric on N (the only possible degeneracy is that the inner product on each fiber may be only semi-definite). To see this, one combines the fact that the differential is a graded group homomorphism and the definition of the metric Δ_x with the previous lemma, concluding that one may recognize $\Delta_x(n_1, n_2)$ by the quantity $d_M(df_x(n_1), df_x(n_2))$. As we shall see in the next section, at points of differentiability when the map is suitably nondegenerate, Δ_x is a well defined Carnot-Carathéodory metric.

2.3. **Jacobians and a weak Sard-like theorem.** Let $f : U \subset N \to M$ be a Lipschitz map between a positive measure subset of a Carnot group and another Carnot group and let k be the Hausdorff dimension of N.

Definition 2.10. *For each $x \in U \subset N$ we define the Jacobian of the map f at x by*

$$J(x) = \lim_{t\to 0} \left\{ \frac{\mathscr{H}^k_M(f(B_N(y,t)))}{\mathscr{H}^k_N(B_N(y,t))} \middle| y \in B_N(x,t) \right\}$$

By arguments analogous to those in [Pau98] concerning the existence of the metric differential, $J(x)$ exists almost everywhere, scales appropriately and is infinitesimally left invariant. Moreover, we now prove that the image of the set where $J(x) = 0$ has $\mathscr{H}^k_M$-measure zero.

Lemma 2.11. *Let $Z = \{x \in U \subset N | J(x) = 0\}$. Then, $\mathscr{H}^k_M(f(Z)) = 0$.*

Proof: Fix $\varepsilon > 0$, $n_0 \in N$ and $0 < R < \infty$. Let $Z_R = Z \cap B_N(n_0, R)$. We will show that $f(Z_R)$ has measure zero. We may cover Z_R by balls $B_N(n,r)$ with the property that $n \in Z_R$ and $\frac{\mathscr{H}^k_M(f(B_N(n,r)))}{\mathscr{H}^k_N(B_N(n,r))} < \varepsilon$. By the assumption that $J(n) = 0$, this covering is fine, so using the Vitali covering lemma, we refine the cover to a countable disjoint collection of balls $\{B_i = B_N(n_i, r_i)\}$ with $r_i < R$ which cover almost all of Z_R. Since f is Lipschitz, we know that $\cup_i f(B_i)$ covers $\mathscr{H}^k_M$ almost all of $f(Z_R)$ as well. Thus

$$\begin{aligned}\mathscr{H}^k_M(f(Z_R)) &\le \sum_i \mathscr{H}^k_M(f(B_N(n_i,r_i))) \\ &< \varepsilon \sum_i \mathscr{H}^k_N(B_N(n_i,r_i)) \\ &= \varepsilon \mathscr{H}^k_N(Z_R) \\ &\le \varepsilon \mathscr{H}^k_N(B_N(n_0,R))\end{aligned}$$

Thus, since ε is arbitrary, $\mathscr{H}^k_M(f(Z_R)) = 0$. The result now follows easily. □

Next we prove a lemma analogous to Sard's theorem concerning the measure of the image of the "degenerate" set. Suppose $f : U \subset N \to M$ is a Lipschitz map and Let $N_x = f(x)df_x(N)$.

Lemma 2.12. *Suppose* $\dim_{\mathscr{H}}(N) \le \dim_{\mathscr{H}}(M)$*. Then, $\mathscr{H}^k_M(\{f(x)|\mathscr{H}^k_M(N_x) = 0\}) = 0$.*

Proof: To begin with, we may assume that at all points Pansu's derivative exists and $J(x)$ exists for all $f(x)$ we are considering since the set of the complementary points has measure zero. First we observe that using the definition of Pansu's differential, the uniformity of its convergence, the left invariance and homothety of the Carnot-Carathéodory metric d_M, we have that $d_M(f(xh_te^v), f(x)h_tdf_x(e^v)) = o(t)$. In particular, this says that $f(B_N(x,t))$ lies in a $o(t)$ neighborhood of $f(x)df_x(B_N(e^0,t))$. Since $f(B)$ has Hausdorff dimension less than or equal to k, we know that $\mathscr{H}^k_M(f(B_N(x,t))) \le \mathscr{H}^k_M(df_x(B_N(x,t)) + o(t^k)$. Now consider an image point $f(x)$ such that $\mathscr{H}^k_M(N_x) = 0$. We discuss separately two cases: $\ker(df_x) = \{e^0\}$ and $\ker(df_x) \ne \{e^0\}$. In the first case, this together with the fact that df_x is a graded group homomorphism implies that the image of df_x is isomorphic to N. Thus, Δ_x is positive definite on N and, since Δ_x is left invariant and admits a homothety,

$df_x : (N, \Delta_x) \to (M, d_M)$ is biLipschitz onto its image and hence cannot have $\mathscr{H}^k_M$-measure zero. So, we may assume that the kernel is nontrivial. Hence, N_x is isomorphic to a quotient of N (by the kernel), call it N'. In particular, realizing N' as a subgroup of N, we see that $\mathscr{H}^k_N(N') = 0$ and so, since df_x is Lipschitz, $\mathscr{H}^k_M(N_x) \leq L^k \mathscr{H}^k_N(N') = 0$. Thus, we have that $\mathscr{H}^k_N(f(B_n(x,t))) = o(t_k)$, yielding the desired result. □

3. An area formula

In this section, we provide a proof of an area formula for Lipschitz maps. A change of variables formula is proved by Vodopyanov and Uhklev ([VU96]) using methods based on Pansu's techniques in [Pan89]. One should be able to extend their arguments to prove the statement below. Also, recently, Magnani ([Mag00]) independently proved the same area formula using a different (but equivalent) definition of Jacobian. Again, let $f : U \subset N \to M$ be a Lipschitz map of a Carnot group N to a Carnot group M. Let k be the Hausdorff dimension of N.

Lemma 3.1. *Fix $\lambda > 1$ and let E be a measurable set in N such that for every $x \in E$, Δ_x exists and is nondegenerate. Then there exists a countable Borel cover of E, $\{B_i\}$ and left invariant CC metrics, d_i, on N such that*

$$\lambda^{-1} d_i(x,y) \leq d_M(f(x), f(y)) \leq \lambda d_i(x,y)$$

for all $x, y \in B_i$. Moreover,

$$\lambda^{-k} \frac{\mathscr{H}^k_i(B_N(x,1))}{\mathscr{H}^k_N(B_N(x,1))} \leq J(x) \leq \lambda^k \frac{\mathscr{H}^k_i(B_N(x,1))}{\mathscr{H}^k_N(B_N(x,1))}$$

for all $x \in B_i$ which are density points of B_i.

Proof: This theorem follows exactly as lemma 4 in [Kir94] which, in turn, follows much of the argument of lemma 3.2.2 in [Fed69]. The metrics d_i are given by $d_i(n_1, n_2) = d_M(df_x(n_1), df_x(n_2))$.

While we explore the concept of N-rectifiability in detail in the next section, we define a set $E \subset M$ to be N-rectifiable if it is the Lipschitz image of a positive measure subset of N.

Corollary 3.2. *If $E \subset M$ is N-rectifiable, then for almost every $x \in E$, $\Theta^k(\mathscr{H}^k_M \lfloor E, x) = \mathscr{H}^k_N(B_N(e^0, 1))$.*

Proof: Using lemma 3.1 above, this follows at density points of the B_i. More precisely, fix $\lambda > 1$ and let B_i and d_i be as in the lemma. Now, for each point of density of B_i which is also a point of Pansu differentiability, let K_i be the preimage of B_i. We have,

$$\lambda^{-2k} \frac{\mathscr{H}^k_N(B_i \cap B_N(x, \frac{\delta}{\lambda}))}{\left(\frac{\delta}{\lambda}\right)^k} \leq \frac{\mathscr{H}^k_M(f(K_i) \cap B_M(f(x), \delta))}{\delta^k}$$

$$\leq \lambda^{2k} \frac{\mathscr{H}^k_N(B_i \cap B_N(x, \delta\lambda))}{(\delta\lambda)^k}$$

Thus, taking advantage of the assumption that x is a density point and letting δ go to zero and λ go to one, we have the density is constant almost everywhere. The result follows. □

Theorem 3. *Suppose $f : N \to M$ is a Lipschitz map. Then, for any $\mathscr{H}_N^k$-measurable set E,*

$$\int_E J(n) d\mathscr{H}_N^k(n) = \int_M N(f|_E, m) d\mathscr{H}_M^k(m)$$

Proof: Assume for a moment that at every point in E, the Pansu differential is well defined and has trivial kernel. In other words, $E \subset \{x | \Delta_x \text{ is a nondegenerate CC metric}\}$. Then, fixing $\lambda > 1$ and using lemma 3.1, we find a countable cover $\{B_i\}$ with the approximative properties described in the lemma. Let $J_i = \frac{\mathscr{H}_i^k(B_N(x,1))}{\mathscr{H}_N^k(B_N(x,1))}$. Note that, under the assumption of nondegeneracy, $\mathscr{H}_N^k$ and $\mathscr{H}_i^k$ both constant multiples of one another and $\mathscr{H}_i^k = J_i \mathscr{H}_N^k$. Using these facts, we have that

$$\lambda^{-k} J_i \leq J(x) \leq \lambda^k J_i \text{ for all } x \in B_i$$

and hence,

$$\begin{aligned}
\lambda^{-2k} \int_{B_i \cap E} J(n) d\mathscr{H}_N^k(n) &\leq \lambda^{-k} \int_{B_i \cap E} J_i d\mathscr{H}_N^k(n) = \lambda^{-k} \int_{B_i \cap E} d\mathscr{H}_i^k \\
&\leq \int_M N(f|_{B_i \cap E}, m) d\mathscr{H}_M^k(m) \leq \lambda^k \int_{B_i \cap E} d\mathscr{H}_i^k \\
&= \lambda^k \int_{B_i \cap E} J_i d\mathscr{H}_N^k(n) \leq \lambda^{2k} \int_{B_i \cap E} J(n) d\mathscr{H}_N^k(n)
\end{aligned}$$

Thus, summing over i and letting $\lambda \to 1$ we have the desired result.

By lemma 2.12 and the arguments used to prove it, we see that at points x of E where Δ_x is not defined or is degenerate, $J(x) = 0$ and the set of all such x maps to a $\mathscr{H}_M^k$ measure zero set, making both sides of the desired equation zero. □.

In light of the discussion above, we note that we could have defined the Jacobian via the Pansu differential as follows: at a point x of Pansu differentiability, $J(x) = \frac{\mathscr{H}_M^k(df_x(B_N(e^0,1)))}{\mathscr{H}_N^k(B_N(e^0,1))}$.

4. CC-rectifiability

Next, we introduce the definitions and basic properties of a theory of rectifiability for subsets of Carnot groups. One should view this analogously to the Euclidean case: Euclidean rectifiable sets are viewed as sets which have, in some sense, a manifold structure while CC-rectifiable sets will have a "manifold" structure where the local geometry modeled by general Carnot groups rather than simply by $\mathbb{R}^n$.

We begin with the relevant definitions. Assume that N and M are Carnot groups and that the Hausdorff dimension of N is k.

Definition 4.1. *Let N be a Carnot group. A subset E of another Carnot Group (M, d_M) is said to be* N-**rectifiable** *if there exists U an positive measure subset of N and a Lipschitz map $f : U \to M$ such that $\mathscr{H}_M^k(E \setminus f(U)) = 0$. E is said to be* **countably** N-**rectifiable** *if there exist a countable number of U_i and $f_i : U_i \to Y$ Lipschitz with $\mathscr{H}_M^k(E \setminus \cup_i f_i(U_i)) = 0$.*

As with the case of $\mathbb{R}^n$-rectifiability, we will be developing the notion of approximate tangent cones and their relation to rectifiability. Next, we wish to consider when a subset of M is well approximated by N. If $i : N \to M$ is a graded group homomorphism, let $N(t) = \{x | d_M(x, i(N)) \leq t\}$.

Definition 4.2. *A subset E of M is N-***approximable** *if for $\mathscr{H}^k_M$ a.e. $a \in E$ we have the following property: if $\alpha > 0$, then there exist an injective homomorphism $i : N \to M$ with $i(e^0) = e^0$ and constants $r_0 > 0, \theta > 0$ such that for any $0 < r < r_0$,*

$$\mathscr{H}^k_M(E \cap B_M(b, \alpha r)) \geq \theta r^k \text{for } b \in a \cdot i(N) \cap B_M(a, r) \tag{1}$$

and

$$\mathscr{H}^k_M(E \cap B_M(a, r) \setminus a \cdot N(\alpha r)) < \alpha r^k \tag{2}$$

Next, we have the first theorem, which follows from the differentiability of Lipschitz maps described in theorem 2.2. The proof of the theorem follows the argument of theorem 15.11 in [Mat95] but with the appropriate changes for the Carnot case. The reader should also note that another key component of the proof relies on the fact that a Carnot group and its tangent cone at a point may be identified due to the existence of a homothety of the Carnot-Carathéodory metric thus allowing the homomorphism of tangent cones to be translated as a statement concerning sets in the group.

Theorem 4. *Every countably N-rectifiable $E \subset M$ with nonzero k-dimensional Hausdorff measure is N-approximable.*

Proof: In this proof, the main information to keep in mind is that the differentiability theory of Lipschitz maps between Carnot groups is close enough to that of Lipschitz maps between Euclidean spaces, allowing many of the Euclidean arguments to be used.

Fix $\varepsilon > 0$. First, we reduce to an easier case. Let $0 < \alpha < 1$. Since E is countably N-rectifiable, we consider an L-Lipschitz map between $E' \subset N$ such that $f(E') \subset E$. To make the reduction to nicer sets, where the lower density is bounded away from zero, we use lemma 3.2.

Using this, we can cover E' up to a set of measure zero by a countable union of subsets, S, with the following properties. For each S, there exists constants $\theta > 0$, $r_0 > 0$ such that for $x \in S$ and $0 < r < r_0$,

$$\mathscr{H}^k_M(f(E') \cap B_M(x, r)) \geq \theta r^k \tag{3}$$

On these sets, we shall verify the properties of N-approximability.

To create a approximating isometric embedded copy of N, we use Pansu's differential mapping. Given a point x at which f is differentiable, Let $i_x(y) = f(x) df_x(x^{-1}y)$ and $N_x = i_x(N)$. Note that the multiplication is multiplication in the Carnot group M. Since df_x is a group homomorphism, we know that N_x is a subgroup of M and from lemma 2.12 we know that for almost every such x, N_x has large k-dimensional Hausdorff measure. In other words, the mapping df_x has, in some sense, full rank. Interpreting this result, we now show that at almost every image point of f, N_x is a isomorphic copy of N. If it were not isomorphic, it would be isomorphic to a subgroup N' of N (since df_x is a graded homomorphism). As a graded subgroup of M, N_x inherits a Carnot-Carathéodory metric which is biLipschitz to the restricted Carnot-Carathéodory metric on N' because all Carnot-Carathéodory metrics given by varying the norm on the distribution of a single nilpotent Lie group are biLipschitz to one another. Thus, N' and N_x have strictly lower Hausdorff dimension than N and so lemma 2.12 shows this may only happen at almost every $f(x)$. Once again, we replace the set E' with a full measure subset of points x such that N_x has "full rank" in the sense described above. Thus, for each x, $(N_x, d_M|_{N_x})$ is biLipschitz to (N, d_N). Denote by $l(x)$ the lower Lipschitz constant for each x. By construction, $l(x) > 0$ for all $x \in E'$.

Recalling that the Lebesgue density theorem holds in N for the measure $\mathscr{H}^k_N$ and using the approximations developed above, we will now find numbers $r_0 > 0$ and $\delta > 0$ and a compact set $E_0 \subset E'$ with $\mathscr{H}^k_N(E' \setminus E_0) < \varepsilon$ consisting only of density points of E' with nice approximative properties. We begin by picking a $\delta < \min\{\frac{\alpha}{4}, \frac{1}{L}\}$. First, pick a compact subset of the density points of E', E_0, so that for $x \in E_0$, $l(x) \geq 2\delta$ (*Property* 1). Second, because N_x approximates $f(E')$ well at $f(x)$ by lemma 2.9, I can pick r_0 small enough so that for $x \in E_0$,

$$d_M(f(y), i_x(y)) < \delta^2 d_N(x, y) \text{ for } y \in B_N(x, r_0) \ (\textit{Property } 2)$$

So far, the choice of r_0 and E_0 depends on δ only. Last, we pick δ and, possibly readjusting r_0 to be smaller still, we may guarantee that for $x \in E_0$, $0 < r < r_0$ and $y \in B_N(x, \frac{r}{\delta})$, $d_N(y, E_0) < \delta^2 r$ (*Property* 3). Property 3 follows from the fact that all points in E_0 are density points of E'.

Now, writing E_0 as the union of finitely many subsets E_i with $diam_N(C_i) < r_0$, we examine each C_i individually. Consider a point $f(x)$ with $x \in C_i$ and $\Theta^k_M(f(E') \setminus f(C_i), f(x)) = 0$. Since almost every point in $f(C_i)$ has this property, we will consider only these points. Let $0 < r < \frac{\delta r_0}{2}$ and pick $i_x(y) \in N_x \cap B_M(f(x), r)$. Note that $y \in B_N(x, \frac{r}{\delta})$ by the property 1. By property 3 above, there exists $z \in E'$ such that $d_N(y, z) < \frac{2r}{\delta}$. Using the triangle inequality, the fact that i_x is L-Lipschitz (recall that $L < \frac{1}{\delta}$) and property 2, we get

$$d_M(f(z), i_x(y)) \leq \delta^2 d_N(x, z) + L d_N(y, z) \leq 3r\delta$$

Since $4\delta < \theta$, we have from equation 3 that

$$\mathscr{H}^k_M(f(E') \cap B_M(i_x(y), \alpha r)) \geq \mathscr{H}^k_M(f(E') \cap B_M(f(z), \delta r)) \geq \theta \delta^k r^k$$

Taking $\lambda = \theta\delta^k$ we have verified equation 1.

To verify equation 2, we observe the following:

First, by property 2,

$$f(C_i \cap B_N(x, \frac{r}{\delta})) \subset N_x(\delta r) \subset N_x(\alpha r)$$

Second, using the lower Lipschitz bound on C_i and property 2 again,

$$f(C_i \setminus B_N(x, \frac{r}{\delta})) \subset M \setminus B_M(f(x), r)$$

and so, $f(C_i) \cap B_M(f(x), r)$ lies inside $N_x(\alpha r)$. Given the density assumption on $f(x)$, that $\Theta^k_M(f(E') \setminus f(C_i), f(x)) = 0$, this implies that, again possibly shrinking r_0, that equation 2 holds. □

5. Measures in cones

As in the $\mathbb{R}^k$-rectifiability theory, we will show that rectifiability can be characterized by approximability (and later by the existence of appropriate tangent spaces). Thus, we will need a method of producing Lipschitz maps. To do so, we follow the idea used in [Mat95] of considering measures of the intersections of cones and the set E to conclude N-rectifiability from N-approximability.

To do this, we need to use a type of projection mapping analogous to the projections onto planes used in the usual theory.

Definition 5.1. *Let V be a vector subspace of $\mathfrak{n}$ and let $< \cdot, \cdot >$ be a Riemannian completion of the CC-inner product on $\mathfrak{n}$ which makes the grading orthogonal. Denote by $v^\perp$ the orthogonal complement of V (with respect to this inner product). Let $\mathfrak{pr}_V : \mathfrak{n} \to V$ by the projection of $\mathfrak{n}$ onto V and let $P_V : N \to e^V$ be the map $exp \circ \mathfrak{pr}_V \circ exp^{-1}$ where exp is the usual exponential map. Also, let Q_V be the map $P_{V^\perp}$. In each of the applications of the projection mappings below, there is an understood base point for the exponential map.*

It is a direct consequence of the equivalence of the Carnot-Carathéodory metric and d_{qn} that the projection defined above is a Lipschitz map if V is a graded Lie subalgebra with compatible grading. Although the Lipschitz constant may not be 1, the map defined above is a projection in the sense that $P_V \circ P_V$ is the identity map. One can easily construct examples of projections which are not Lipschitz.

Example: Consider the 3-dimensional Heisenberg group, H^3, with a left invariant Carnot-Carathéodory metric, d_{cc}. Let V be the vector subspace spanned by the "Z" (non-distributional) direction. Then, the exponential image of V in H^3 is a 1-parameter subgroup but V is not a Lie subalgebra with compatible grading. We will now directly show that this projection is not Lipschitz. Not surprisingly, this stems from the nontrivial bracket structure. Giving H^3 coordinates $\{X, Y, Z\}$, we describe an element $e^{aX+bY+cZ}$ by the triple (a, b, c). Then $d_{cc}((a, b, c), (\alpha, \beta, \gamma)) = d_{cc}((0, 0, 0), (\alpha - a, \beta - b, \gamma - c + \frac{1}{2}(\alpha b - a\beta)))$. Fixing $\varepsilon > 0$ and taking $\alpha = a + \varepsilon$, $\beta = b + \varepsilon$ and $a = \frac{(\gamma - c) - \frac{b\varepsilon}{2}}{\varepsilon}$, this simplifies to $d_{cc}((0, 0, 0), (\varepsilon, \varepsilon, 0)) = \varepsilon d_{cc}((0, 0, 0), (1, 1, 0))$. Computing the distance under the image of P_v, we have

$$d_{cc}(P_V((a, b, c)), P_V((\alpha, \beta, \gamma))) = d_{cc}((0, 0, c), (0, 0, \gamma)) = \sqrt{|\gamma - c|} d_{cc}((0, 0, 0), (0, 0, 1))$$

Thus, since ε, γ and c are arbitrary, we see that the map cannot be Lipschitz. □

Definition 5.2. *Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{m}$ suppose G is the orthogonal complement of $\mathfrak{n}$. If $m_0 \in M$, $0 < s < 1$, and $0 < R < \infty$, we define*

$$X(n_0, G, s) = \{m_1 \in M | d_M(Q_G(m_1), Q_G(m_0)) < s d_M(m_1, m_0)\}$$

Moreover, let

$$X(m, R, G, s) = X(m, G, s) \cap B_M(m_0, R)$$

The reader should look closely at this definitions of "cones". While they are the same in form as the Euclidean versions used by Mattila in [Mat95], the actual objects look somewhat different in practice (it is useful and relatively easy to investigate the shape of these objects in the Heisenberg group). While they still have the same general form of an "X" emminating from the base point, the spreading of the limbs of the "X" is no longer quite linear. In practice, we will see that this will make no difference for our applications.

We begin with a simple lemma to aid in producing Lipschitz maps to rectifiable sets.

Lemma 5.3. *Suppose $Y \subset M$ and G is the orthogonal complement of a graded Lie subalgebra $\mathfrak{n}$ of $\mathfrak{m}$, $0 < s < 1$ and $0 < R < \infty$ then if $Y \cap X(y, R, G, s) = \emptyset$ for all $y \in Y$, then Y is N-rectifiable.*

Proof: Geometrically, this says roughly that the Lie algebra preimage of the set Y at each point locally lies in $\mathfrak{n}$. Thus it makes sense that one can approximate the set by Lipschitz images of pieces of N. To prove this, we first may assume that $diam_M(Y) < R$, otherwise we may simply cut up Y into a countable number of such pieces. Next, we observe that if $y_1, y_2 \in Y$, the hypothesis of empty intersection implies that $d_M(Q_G(y_1), Q_G(y_2)) > s d_M(y_1, y_2)$. Thus, the map $(Q_G|_Y)^{-1}$ is Lipschitz with constant less than $\frac{1}{s}$. Since $Q_G|_Y$ and $(Q_G|_Y)^{-1}$ are both Lipschitz , we see that there exists $\overline{Y} \subset N$, a positive measure subset such that $(Q_G|_Y)^{-1} : \overline{Y} \to Y$. Hence, Y is N-rectifiable. $\square$

The reader should note that one implication of the lemma is that a purely N-unrectifiable set must have a nonempty intersection for almost all points in the set.

Next, we refine the lemma above, replacing the requirement of empty intersection with a bound on the measure of the intersection.

Lemma 5.4. *Let $G, \mathfrak{n}$ and s be as above. Let $0 < \delta < \infty$ and $0 < \lambda < \infty$. Let k be the Hausdorff dimension of N. If $Y \subset M$ is purely N-unrectifiable and $\mathscr{H}^k_M(Y \cap X(y, r, G, s)) \leq \lambda r^k s^k$ for $y \in Y$ and $0 < r < \delta$, then $\mathscr{H}^k_M(Y \cap B_M(m, \frac{\delta}{6})) \leq C\lambda\delta^k$ for all $m \in M$ where C is a constant depending only on k.*

Proof: This proof is exactly the same as the proof of lemma 15.14 in [Mat95]. The constant C is given by $2 \cdot 20^k \cdot \mathscr{H}^k_N(B_N(e^0, 1))$.

Corollary 5.5. *Let $G, \mathfrak{n}$ and s be as in the last lemma. Suppose $Y \subset M$ is purely N-unrectifiable with $\mathscr{H}^k_M(Y) < \infty$, then there exists a constant C depending only on k and $\mathscr{H}^k_N(B_N(e^0, 1))$ such that*

$$\Theta^{*k}(Y \cap X(y, G, s), y) \geq C s^k$$

for H^k_M a.e. $y \in Y$.

Proof: Again, this is the same as the proof of corollary 15.16 in [Mat95]. The only detail which must be checked is the lower bound on the density of sets of positive measure at almost ever point. This is covered in lemma 2.5.

6. Characterization of Rectifiability

As in the previous section, we assume that N and M are Carnot groups of Hausdorff dimensions k and l respectively.

Definition 6.1. *Suppose $Y \subset M$, $m \in M$ and V is a subspace of $\mathfrak{m}$ with $exp(V)$ isomorphic to N. We say that $exp_m(V)$ is an approximate tangent cone for Y at m if*

$\Theta^{*k}(Y,m) > 0$ *and for all* $0 < s < 1$,

$$\lim_{r \to 0^+} \frac{\mathscr{H}^k_M(Y \cap B_M(m,r) \setminus X(m,V,s))}{r^k} = 0$$

Following the Euclidean notation, we denote the set of all approximate tangent cones as $apTan^N(Y,m)$. *Note that this set depends on a choice of Carnot group* N.

Next, we come to the first main theorem; we characterize N-rectifiability in terms of N-approximability and the existence of approximate tangent cones almost everywhere.

Theorem 5. *Let* N *and* M *be Carnot groups of Hausdorff dimensions* k *and* l *respectively. Let* Y *be a* $\mathscr{H}^k_M$*-measurable subset of* M *with* $\mathscr{H}^k_M(Y) < \infty$. *Then, the following are equivalent:*

1. Y *is* N*-rectifiable.*
2. Y *is* N*-approximable.*
3. *For* $\mathscr{H}^k_M$ *almost all* $y \in Y$, *there is a unique approximate tangent cone at* y *isometric to* N.
4. *For* $\mathscr{H}^k_M$ *almost all* $y \in Y$, *there exists some approximate tangent cone at* y *isometric to* N.

Proof: As with the previous results, this proof follows the development of [Mat95] with changes for the Carnot case.
(1) $\Rightarrow$ (2) is theorem 4 and (3) $\Rightarrow$ (4) is trivial, so it remains to prove the other two implications.
(2) $\Rightarrow$ (3)
Notice that, for $\varepsilon > 0$ sufficiently small, $B_M(y,r) \setminus X(y,N,s) \subset (B_M(y,r) \setminus N(\varepsilon sr)) \cup B_M(y,\varepsilon r)$. From this, and the assumption that Y is N-approximable (use the second property in the definition), we see that a copy of N is an approximate tangent cone for Y at almost every point. To see that the approximate tangent cone is unique, we use property (1) in definition 4.2 and the definition of approximate tangent cone.
(4) $\Rightarrow$ (1)
We will use lemma 5.4 to show this implication via contradiction. Thus, we assume Y is purely N-unrectifiable and we will show that it can possess an approximate tangent cone isomorphic to N almost nowhere. Consider M as $\mathbb{R}^n$ with the standard Euclidean metric and let E_V denote projection onto V, a linear subspace. Let k_0 be the topological dimension of N. Let $m \in \mathbb{N}$ (we will adjust the choice of m later in the proof). We can use the compactness of $G(n,k_0)$ (using the standard operator norm) to cover $G(n,k_0)$ by finitely many balls of radius $\frac{1}{2m}$. Call this family of balls $\mathcal{B}$. Consider now the subset $\mathcal{G} = \{B_1, ..., B_l\}$ of $\mathcal{B}$ of balls containing k_0-planes that, under the induced grading from N, are isomorphic to N. Let $G_{gr}(n,k_0) \subset G(n,k_0)$ be the subset of k_0-planes isomorphic to N under the induced grading. Now, for each B_i, pick $W_i \in B_i \cap G_{gr}(n,k_0)$. Then, by construction, for $v \in B_i \cap G_{gr}(n,k_0)$, we know that $||E_V - E_{W_i}|| < \frac{1}{m}$. In terms of the CC projection, using the equivalence of d_M and d_{qn}, if α is any vector, we have

$$d_M(P_V(e^\alpha), P_{W_i}(e^\alpha)) \leq C(m) d_M(e^0, e^\alpha)^\kappa$$

where $C(m) = const \cdot (\frac{1}{m})^\kappa$ and κ is an appropriate power determined by the grading and CC metric on M. Here, we abuse notation and assume V and W_i are exponentiated at the same base point.

Fixing W_i, let $B = \left\{ y \in Y \middle| \exists V \in B_i) \text{ s.t. } V \in apTan^N(Y, y) \right\}$. We want to show that B has $\mathscr{H}^k_M$-measure zero. Suppose, on the contrary, that B has positive measure. Since, by assumption, we know that every point in B has an approximate tangent plane in B_i, given $\lambda > 0$, $\exists r_0 > 0$ so that

$$C = \{b \in B | \sup_{0<r<r_0} \frac{\mathscr{H}^k_M(B \cap B_M(b,r) \setminus X(b, V, C(m)))}{r^k} < \lambda C(m)^k\}$$

has positive $\mathscr{H}^k_M$-measure.

Next, we claim that, for sufficiently large $m \in \mathbb{N}$,

$$X(b, r, W_i^\perp, C(m)) \subset B_M(b,r) \setminus X(b, V, C(m))$$

Geometrically, this is almost clear, but we shall prove it anyway. Suppose the claim does not hold; for every m, there exists $y \in X(b, r, W_i^\perp, C(m)) \cap X(b, V, C(m))$ and $y \neq b$. From the definitions of the "X" sets, we have

$$(i) \qquad d_M(P_W(y), b) \leq C(m) d_M(y, b)$$

$$(ii) \qquad d_M(P_{V^\perp}(y), b) \leq C(m) d_M(y, b)$$

From above, we know $d_M(P_w(y), P_V(y)) \leq C(m) d_M(b, y)^\kappa$. Using the triangle inequality and (i), we have

$$(iii) \qquad d_M(P_V(y), b) \leq C(m) d_M(y, b) + C(m) d_M(y, b)^\kappa$$

Now, as $m \to \infty$, $C(m) \to 0$ so, considering (ii) and (iii) and using the equivalence of d_m and d_{qn}, no such y can exist thus proving the claim.

Picking m so that the claim is true, we can now finish the proof of this implication using the following computation:

$$\begin{aligned} \mathscr{H}^k_M(C \cap X(b, r, W_i^\perp, C(m))) &\leq \mathscr{H}^k_M(C \cap (B_M(b,r) \setminus X(b, V, C(m)))) \\ &\leq \lambda C(m)^k r^k \end{aligned}$$

Picking λ sufficiently small, we violate lemma 5.5. Thus, C has measure zero as does B proving the implication (4) $\Rightarrow$ (1). $\square$

Remarks:

- We reiterate that the class of N-rectifiable manifolds, for a Carnot group N, has good local properties. In particular, the tangent cone at each point (in this setting, it is easy to see using the N-approximability at a point that the unique approximate tangent cone will coincide with the tangent cone in the sense of Gromov) is isometric to N in contrast to the examples cited earlier. This should allow for a stronger analysis of the local geometry of N-rectifiable smooth submanifolds.

- How large is the set of N-rectifiable smooth manifolds? In particular, which Carnot-Carathéodory manifolds are realized as N-rectifiable sets? One can see that every compact contact Carnot-Carathéodory manifold, C, (using the contact structure to define the subbundle) is N-rectifiable for a suitable N. One sees this by first using D'Ambra's isometric embedding theorem for contact CC manifolds ([D'A95]) to realize C as a submanifolds of the Heisenberg group of large enough dimension. Then, calculating the tangent cone at each point (using the contact form and the method of Bellaiche ([Bel96]) or Mitchell ([Mit85])), one can show that C has an approximate tangent cone at each point isometric to a suitable N. Appealing to the theorem above yields N-rectifiability.

7. Level Sets of Lipschitz Functions

Although the last section proved the equivalence of N-rectifiability, N-approximability and existence/uniqueness of approximate tangent cones along the same lines as the Euclidean theory, we note that projections onto subgroups of a Carnot groups are not necessarily Lipschitz. For example, as shown in section 5, the projection of H^3 onto the "yz" subgroup is not Lipschitz. The failure of the projections to be Lipschitz is related to the compatibility of the induced grading on the subgroup. In this section, we investigate the properties of level sets of Lipschitz functions on Heisenberg groups and, we will see, that in modeling the local behavior of such sets, we will need to use subgroups of the Heisenberg group onto which the projection mapping is not Lipschitz. In fact, as in the Euclidean case, the subgroup which locally models $f^{-1}(t)$ at a point x is $ker(df_x)$ which, by Pansu's differentiability theory, exists almost everywhere and is a subgroup for a.e. x.

For this section, we assume N is the Heisenberg group of dimension $2n+1$ and $k = 2n+2$ is the Hausdorff dimension of N. We begin with an example.

Example: Consider the mapping $f : H^3 \to \mathbb{R}$ given by $e^{aX+bY+cZ} \to \sqrt{a^2+b^2+|c|}$. This is a Lipschitz map (any quasi-norm on H^3 is biLipschitz equivalent to the Carnot-Carathéodory metric) and the inverse image of $t \in \mathbb{R}$ is the boundary of $B_{qn}(n_0, t)$. We will now calculate the kernels of the differential mappings at each point. First, we calculate $df_x(e^{aX+bY+cZ})$. Fix $x = e^{\alpha X+\beta Y+\gamma Z}$. Since,

$$xh_r e^{aX+bY+cZ} = e^{(\alpha+ra)X+(\beta+rb)Y+(\gamma+r^2c+\frac{r}{2}(\alpha b-a\beta))Z}$$

we have

$$\begin{aligned}
&\lim_{r\to 0^+} \frac{\sqrt{(\alpha+ra)^2+(\beta+rb)^2+|\gamma+r^2c+\frac{r}{2}(\alpha b-a\beta)|}-\sqrt{\alpha^2+\beta^2+|\gamma|}}{r} \\
&= \lim_{r\to 0^+} \pm\frac{r\alpha+r^2a+r\beta+r^2b+rc+\frac{1}{4}(\alpha b-a\beta)}{\sqrt{(\alpha+ra)^2+(\beta+rb)^2+|\gamma+r^2c+\frac{r}{2}(\alpha b-a\beta)|}} \\
&= \pm\frac{\alpha b-a\beta}{4\sqrt{\alpha^2+\beta^2+|\gamma|}}
\end{aligned}$$

So, for this fixed x, we see that, so long as $\alpha \neq 0$,

$$ker(df_x) = \{e^{aX+bY+cZ} | b = \frac{\beta}{\alpha}a\}$$

Note that these subgroups are all mutually isomorphic. For $\alpha = 0$, one can easily calculate the remaining cases. One should note that the case when $\alpha = \beta = 0$ is particularly interesting, for at those points, the kernel is $\{e^{aX+bY+cZ}\}$. These are points where Pansu's differentiability theory fails.

In this section, we are concerned with rectifiability properties of level sets of Lipschitz functions. The most natural candidate for a Lipschitz map between $U \subset N$ and $f^{-1}(t)$ would be, as in the Euclidean case, the inverse of the projection of $f^{-1}(t)$ to $ker(df_x)$. Unfortunately, this map is not Lipschitz. We show this in a special case. Consider $x = e^X$ in $f^{-1}(1)$. Then, by the above calculation, $ker(df_x) = \{e^{bY+cZ}\}$. Let $P_x : f^{-1}(1) \to ker(df_x)$ be the map $P_x(e^{aX+bY+cZ}) = e^{bY+cZ}$ where $|a^2 + b^2 + |c|| = 1$. Taking $n_1 = e^{aX+(1-a^2)Z}$ and $n_2 = e^{aX+\varepsilon Y+(1-a^2-\varepsilon^2)Z}$ for $\varepsilon > 0$ small and a close to 1, we have

$$\begin{aligned} d_{H^3}(n_1, n_2) &= d_{H^3}(e^0, e^{\varepsilon Y+(a\varepsilon-\varepsilon^2)Z}) \\ &\geq C_1 a \varepsilon^{\frac{1}{2}} \end{aligned}$$

But, computing the distance between the projections, we have

$$\begin{aligned} d_{H^3}(P_x(n_1), P_x(n_2)) &= d_{H^3}(e^0, e^{\varepsilon Y-\varepsilon^2 Z}) \\ &\leq C_2 \varepsilon \sqrt{2} \end{aligned}$$

Taking the quotient and applying the above estimates yields that $Lip(P_x) \geq C_3 \varepsilon^{-\frac{1}{2}}$. Letting ε go to zero shows that the inverse map cannot be Lipschitz. □

In the theory of $\mathbb{R}^n$-rectifiability, one has a nice decomposition of Lipschitz maps between Euclidean spaces controlled by the area and co-area formulae. In particular, one can conclude that the inverse image of a point under a Lipschitz map is $\mathbb{R}^k$-rectifiable. We prove, in this section, that level sets of Lipschitz maps from the Heisenberg group to $\mathbb{R}$ are locally N'-approximable and have approximate tangent cones isomorphic to N' (locally) a.e. for N' some subgroup of N. The proof of such a fact is significantly harder than in the Euclidean case because many of the tools (even the ones used in the previous sections) are not available in the cases where the local model for the level sets are subgroups of Carnot groups with incompatible gradings. To circumvent the standard arguments, we employ an approximation of the Lipschitz function by a smooth function using the usual mollification procedure. This gives us two tools - the smooth approximation and the Pansu differential of the function. Combining these two different perspectives allows us the desired control.

Lemma 7.1. *If $f : N \to \mathbb{R}$ is a Lipschitz map, there exist smooth functions $f_i : N \to \mathbb{R}$ which converge uniformly to f on compact sets. In addition, if X is a horizontal vector field on N, then $Xf_i \to Xf$ uniformly on compact sets where Xf is defined as well.*

Remarks on proof: This is a simple consequence of the convergence properties of convolution operators on graded nilpotent Lie groups and the usual mollification and approximation procedures. As in the Euclidean case, one has an approximation continuous functions by smooth functions which converge uniformly on compact sets. These facts are well known and can be found in, for example, [FS82]. The reader should also see [GN96], [FSSC95] and [FSSC99] (in particular proposition 5.8 and theorem 6.4) for an explicit discussion of mollifiers acting on functions on Carnot-Carathéodory spaces.

Lemma 7.2. *For almost every $t \in \mathbb{R}$, $f_i^{-1}(t)$ are all smooth submanifolds of N.*

Proof: Standard.

Lemma 7.3. *For a.e. $t \in \mathbb{R}$, almost every point of $f_i^{-1}(t)$ is a point of Pansu differentiability for f. For almost every $t \in \mathbb{R}$, $f^{-1}(t)$ is $\mathscr{H}_N^{k-1}$ measurable and, for such a t, almost every $x \in f^{-1}(t)$ is a point of Pansu differentiability for f.*

Proof: The first statement is a consequence of the coarea formula for smooth real valued maps on Carnot groups. See proposition 2.6 and [Hei95] section 4.7. The second statement follows directly from lemma 2.10.15 in [Fed69].□

Lemma 7.4. *Let Ω be a compact set in N and let $\Omega_t = \Omega \cap f^{-1}(t)$. Then, for almost every t and for almost every $x \in \Omega_t$, there exists $r_x > 0$ such that for a.e. $y \in B_N(x, r_x) \cap \Omega_t$, $ker(df_y)$ and $ker(df_x)$ are isomorphic as subgroups of N.*

Proof: Let m be the topological dimension of N. Pick t so that $f_i^{-1}(t)$ are smooth, a.e. point of each $f_i^{-1}(t)$ is a point of Pansu differentiability for f and almost every point of $f^{-1}(t)$ is a point of Pansu differentiability for f. The exponential preimage of $ker(df_x)$ in the Lie algebra based at x is a vector subspace of $\mathfrak{n}$. Denote this subspace by V_x. By the induced grading data on V_x, we mean the extra data that the grading imposes on V_x by restriction (the terminology is, at best, misleading since the induced grading data is rarely a grading on V_x). Now, the subgroups $ker(df_x)$ and $ker(df_y)$ are isomorphic and biLipschitz with respect to the quasi-norm of N restricted to the respective subgroups if V_x and V_y have compatible induced grading data, i.e. if there is a algebraic homomorphism between V_x and V_y which respects the induced grading data. This is an open condition on $m-1$ planes in $\mathbb{R}^m$ and so, since df_x is continuous on the full measure subset of Ω_t where it is defined, the set $\{y \in \Omega_t | ker(df_y) \equiv ker(df_x)\}$ is relatively open. Moreover, by iterating this procedure (if the complement of this set in Ω_t has interior), we quickly see that for a.e. point $x \in \Omega_t$, $\exists r_x > 0$ s.t for $y \in B_N(x, r) \cap \Omega_t$, $ker(df_y) = ker(df_x)$. □

Theorem 6. *Let G be the Heisenberg group of dimension $2n+1$ and let $f : G \to \mathbb{R}$ be a Lipschitz map. Then, for a.e. $t \in \mathbb{R}$ and a.e. $x \in f^{-1}(t)$, there exists a subgroup $T_x \subset G$ and $r_x > 0$ such that $B_N(x, r_x) \cap f^{-1}(t)$ is T_x-approximable and $f^{-1}(t)$ has a unique approximate tangent cone isomorphic to T_x for a.e. $x \in B_N(x, r_x) \cap f^{-1}(t)$.*

The proof of this theorem follows from the next two lemmas and lemma 7.4.

Lemma 7.5. *Suppose $x \in f^{-1}(t)$ is a point of Pansu differentiability for f. If $x' \in f^{-1}(t) \cap B_N(x, s)$ then $d_N(x', T_x) = O(s^2)$.*

Proof: Consider a one parameter family of points $xh_s e^{w_1(s)+w_2(s)} \in f^{-1}(t)$ where $w_1(s) \in ker(df_x)$, $w_2(s)$ is perpendicular to the kernel and $d_N(e^0, e^{w_1(s)+w_2(s)}) = 1$ for all s. Now, by the choice of these points $|f(xh_s e^{w_1(s)+w_2(s)}) - f(x)| = 0$. Moreover, using the Pansu differentiability at the point x, $|f(xh_s e^{w_1(s)+w_2(s)}) - (f(x) + s df_x(e^{w_2(s)})| = o(s)$ and so, using the left invariance and homothety in the target, we have:

$$d_M(df_x(e^{w_2(s)}), e^0) = O(1)$$

Moreover, we notice that $df_x|_{N/ker(df_x)}$ is biLipschitz since $d_N|_{N/ker(df_x)}$ and $d_{\mathbb{R}}|_{Im(df_x)}$ are simply nondegenerate metrics on $\mathbb{R}$ and df_x is a Lipschitz group homomorphism respecting dilations. Using this, we have that

$$d_N(e^{w_2(s)}, e^0) = O(1)$$

So, using the Campbell-Baker-Hausdorff formula and the fact that $w_2(s)$ lies in a Lie subalgebra, $d_N(xh_s e^{w_1(s)+w_2(s)}, ker(df_x)) = d_N(h_s e^{w_2(s)}, e^0) = o(s)$. $\square$

Lemma 7.6. *Fix a compact set $\Omega \subset N$. Let $C = f^{-1}(t) \cap \Omega$ and $C_i = f_i^{-1}(t) \cap \Omega$. Then, there exists a constant $\kappa > 0$ such that for $x \in N$ and $r > 0$,*

$$\kappa \mathscr{H}_N^{k-1}(C \cap B_N(x,r)) \geq \overline{\lim}_{i\to\infty} \mathscr{H}_N^{k-1}(C_i \cap B_N(x,r))$$

Proof: This is not a hard fact to prove, but we cite it as a special case of lemma 8.35 in [DS97]. First we check whether each $C_i \cap B_N(x,r)$ is Ahlfors subregular of dimension $k-1$, i.e. if there exists a constant K_i such that $K_i^{-1} s^{k-1} \leq \mathscr{H}_N^{k-1}((C_i \cap B_N(x,r)) \cap B_N(y,s)) \leq K_i s^{k-1}$ for $y \in C_i \cap B_N(x,r)$ and $s > 0$. However, since the f_i are smooth, lemma 2.7 provides a method for computing the Hausdorff measure. We notice that Xf_i and Xf_j are close to one another (up to a set of negligible measure) for i,j sufficiently large and for any smooth vector field X. Note that, if Z is the vector field pointing in the nonsubbundle direction in N, it is entirely possible the Zf does not exists, which would cause $|Zf_i| \to \infty$ as $i \to \infty$. Now, via a direct and straightforward computation using the implicit function theorem and multivariate calculus and proposition 2.6, we find for $x \in N$ and $r > 0$,

$$\mathscr{H}_N^{k-1}(C_i \cap B_n(x,r)) = \int_{C_i \cap B_n(x,r)} \frac{|\nabla_0 f_i|}{|\nabla f_i|} dA \leq K_i r^{k-1}$$

One should notice that, due to the previous remark, $|\nabla f_i|$ may become arbitrarily large as $i \to \infty$. However, in computing the Jacobian factor for the area integral, exactly the same potentially unbounded factor appears, leading to a finite integral. Note also that K_i depends only on the structure of the Heisenberg group and the function f. Moreover, since f_i and f_j are C^1 close for sufficiently large i,j, we see that the K_i are uniformly bounded. Thus, the hypotheses of lemma 8.35 in [DS97] are satisfied. $\square$

Lemma 7.7. *Fix $\alpha > 0$. Then there exists $s_0 > 0$, $\theta > 0$ such that for $0 < s < s_0$ and $x' \in T_x \cap B_N(x,s)$ then, $\mathscr{H}_N^{k-1}(f^{-1}(t) \cap B_N(x', \alpha s)) \geq \theta s^{k-1}$.*

Proof: To determine s_0, we use lemma 7.5. The estimate in this lemma and the triangle inequality tell us that if $x' \in T_x \cap B_N(x,s)$ then $d_N(x', f^{-1}(t)) = o(s)$. Picking s_0 small enough (this will depend on α), we can guarantee that $B_N(x', \frac{\alpha s}{2}) \cap f^{-1}(t) \neq \emptyset$ for $0 < s < s_0$ and $x' \in T_x \cap B_N(x,s)$. Let $B_s^i = B_N(x', \alpha s) \cap f_i^{-1}(t)$. We know from the construction of the f_i that the B_s^i converge to $B_s = B_N(x', \alpha s) \cap f^{-1}(t)$. Therefore, by lemma 7.6, $\overline{\lim}_{i\to\infty} \mathscr{H}_N^{k-1}(B_s^i) \leq \kappa \mathscr{H}_N^{k-1}(B_s)$. Also, in the proof of lemma 7.6, we noted that $f^{-1}(t) \cap \Omega$ is Ahlfors regular. By our choice of s_0, we know that, for sufficiently large i, there are points $x_i \in f_i^{-1}(t) \cap \Omega$ such that $B_N(x_i, \frac{\alpha s}{4}) \cap f_i^{-1}(t) \subset B_s^i$. By Ahlfors regularity, we have $\mathscr{H}_N^{k-1}(B_s^i) \geq \frac{K\alpha^{k-1}s^{-k-1}}{4^{k-1}}$, yielding the desired estimate. $\square$

Proof of theorem 6: First, lemma 7.4 gives the constant r_x in the theorem and the local structure for a.e. point in the ball. Next, for this candidate local structure, we must verify conditions (1) and (2) in definition 4.2. Condition (1) is verified by lemma 7.7. Moreover, a consequence of lemma 7.5, there exists $r_0 > 0$ such that $(f^{-1}(t) \cap B_N(x,r)) \setminus T_x(\alpha r) = \emptyset$ for $0 < r < r_0$. The proof of the existence of a unique approximate tangent cone isometric to N follows just as in one implication of theorem 5. $\square$

Remark: The techniques and computations in this section are very similar in spirit to those used to prove the implicit function theorem (on the Heisenberg group) in [FSSC99]. It is a natural question as to whether the results above and the results ion [FSSC99] can be extended to Carnot groups other than the Heisenberg group. The main technical difficulty in extending the above arguments (which is also evident in [FSSC99]) is generalizing lemma 7.6 in which a precise cancellation occurs allowing the proof to go through. In more general Carnot groups, this cancellation would almost certainly not hold for such approximations to generic Lipschitz maps.

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Rice University, Houston, TX, 77005
E-mail address: pauls@math.rice.edu