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Simplicity of Ultragraph Algebras

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ABSTRACT. In this paper we analyze the structure of C^*-algebras associated to ultragraphs, which are generalizations of directed graphs. We characterize the simple ultragraph algebras as well as deduce necessary and sufficient conditions for an ultragraph algebra to be purely infinite and to be AF. Using these techniques we also produce an example of an ultragraph algebra that is neither a graph algebra nor an Exel-Laca algebra. We conclude by proving that the C^*-algebras of ultragraphs with no sinks are Cuntz-Pimsner algebras.

1. INTRODUCTION

In [23] a generalization of a directed graph, called an ultragraph, was defined. In analogy with the C^*-algebras of directed graphs, it was also shown how to associate a C^*-algebra $C^*(\mathcal{G})$ to an ultragraph $\mathcal{G}$. These ultragraph algebras include the C^*-algebras of graphs [14, 15, 1, 9] as well as the Exel-Laca algebras of [7]. Furthermore, it was shown that many of the techniques used for graph algebras can be applied to obtain similar results for ultragraph algebras. This has many important consequences. First, one can now study Exel-Laca algebras in terms of ultragraphs. Thus the frequently complicated and cumbersome matrix manipulations involved in studying Exel-Laca algebras may be replaced by graphical techniques that are often easier to deal with as well as more visual. In addition, since the classes of graph algebras and Exel-Laca algebras each contain C^*-algebras that are not in the other, similar results concerning the two classes have often had to be proven separately for each class. Because ultragraph algebras contain both of these classes, they provide a context in which these similar results can be proven once and then applied to the special cases of graph algebras and Exel-Laca algebras.

In this paper we build upon the work in [23] and analyze the structure of $C^*(\mathcal{G})$. Throughout we have two goals. First, we wish to show that graph algebra techniques can be used to obtain many results concerning $C^*(\mathcal{G})$ and that many properties of $C^*(\mathcal{G})$ can be read off from the ultragraph $\mathcal{G}$. Second, we wish to convince the reader that the ultragraph approach provides a more convenient method for studying Exel-Laca algebras. In their seminal paper [7], Exel and Laca describe how to associate a graph $\mathrm{Gr}(A)$ to a $\{0,1\}$-matrix A. Throughout their analysis many conditions are stated in terms of the graph $\mathrm{Gr}(A)$ and it is shown that certain properties of $\mathcal{O}_A$ are reflected in $\mathrm{Gr}(A)$. As in [23] we shall associate an ultragraph $\mathcal{G}_A$ to A for which $C^*(\mathcal{G}_A)$ is canonically isomorphic to $\mathcal{O}_A$. We shall show that the ultragraph $\mathcal{G}_A$ provides much of the same information as $\mathrm{Gr}(A)$, and in addition

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there are aspects of $\mathcal{O}_A$ that can be easily obtained from $\mathcal{G}_A$ but not from $\mathrm{Gr}(A)$. In particular, we examine how the simplicity of $\mathcal{O}_A$ is reflected in $\mathcal{G}_A$.

After some preliminaries, we begin in §3 by considering the ideals of $C^*(\mathcal{G})$ and determining necessary and sufficient conditions for $C^*(\mathcal{G})$ to be simple. Finding conditions for simplicity in graph algebras and Exel-Laca algebras has been an elusive goal of many authors in the past few years. It was not until recently that such conditions were obtained, and the preliminary work involved many partial results as well as high-powered techniques and sophisticated tools. Building on the simplicity criteria for Cuntz-Krieger algebras [3, Theorem 2.14] conditions for simplicity of C^*-algebras of certain graphs were obtained in [14, Corollary 6.8] and similar results for row-finite graphs were obtained in [1, Proposition 5.1]. In [9] C^*-algebras of arbitrary (i.e., not necessarily row-finite) graphs were introduced and it was shown that transitivity of the graph was a sufficient (but not necessary) condition for simplicity of the C^*-algebra [9, Theorem 3]. In [11, Corollary 4.5] it was shown that for graphs in which every vertex emits infinitely many edges, transitivity was also a necessary condition for simplicity. In addition, Exel and Laca gave sufficient conditions for simplicity of the Exel-Laca algebras in [7, Theorem 14.1]. Necessary and sufficient conditions for simplicity of Exel-Laca algebras were finally obtained by Szymański in [22, Theorem 8] and his result could be adapted to give necessary and sufficient conditions for simplicity of C^*-algebras of arbitrary graphs [22, Theorem 12]. His conditions for the Exel-Laca algebras $\mathcal{O}_A$ were stated in terms of saturated hereditary subsets of the index set of A, and his conditions for graph algebras were stated in terms of saturated hereditary subsets of the graph's vertices. Shortly afterwards independent results of [17, Theorem 4] and [4, Corollary 2.14] also gave necessary and sufficient conditions for simplicity of graph algebras in terms of reachability of certain vertices in the graph.

In this paper we give necessary and sufficient conditions for an ultragraph algebra to be simple. We state this result in two ways. In Theorem 3.10 we give the result in terms of saturated hereditary subcollections, and as one would expect the result is very much like that of Szymański's in [22, Theorem 12]. In addition, in Theorem 3.11 we give a characterization of simplicity in terms of reachability of certain vertices. Although this result contains [17, Theorem 4] and [4, Corollary 2.14] as special cases, it is a much less obvious generalization. We conclude §3 with an example showing that the ultragraph $\mathcal{G}_A$ is a better tool than the graph $\mathrm{Gr}(A)$ for determining the simplicity of the Exel-Laca algebra $\mathcal{O}_A$.

In §4 we give necessary and sufficient conditions for $C^*(\mathcal{G})$ to be purely infinite and to be AF. These conditions are stated in terms of the ultragraph $\mathcal{G}$ and show that, as with graph algebras, the structure of $C^*(\mathcal{G})$ is reflected in $\mathcal{G}$. Using our results from the previous section we also show that the dichotomy of simple graph algebras holds for simple ultragraph algebras; that is, every simple ultragraph algebra is either AF or purely infinite.

In §5 we use the techniques developed in our analysis of ideals in §3 to produce an ultragraph algebra that is neither an Exel-Laca algebra nor a graph algebra. This result is important because it shows that the class of ultragraph algebras is larger than the graph algebras and the Exel-Laca algebras. Hence our results in this paper and the results of [23] are seen to be more substantial since they hold for C^*-algebras other than just the graph algebras and Exel-Laca algebras.

We conclude in §6 by showing that the C^*-algebras of ultragraphs with no sinks may be realized as Cuntz-Pimsner algebras. There is currently much interest in Cuntz-Pimsner algebras, and since ultragraph algebras are contained in this class it is possible that they could serve as interesting examples and perhaps provide greater insight into the study of general Cuntz-Pimsner algebras.

2. Ultragraph Algebras

In this section we review the basic definitions and properties of ultragraphs and their C^*-algebras. For a more thorough introduction, we refer the reader to [23].

Definition 2.1. An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to P(G^0)$, where $P(G^0)$ denotes the collection of nonempty subsets of G^0.

If $\mathcal{G}$ is an ultragraph, then a vertex $v \in G^0$ is called a *sink* if $|s^{-1}(v)| = 0$ and an *infinite emitter* if $|s^{-1}(v)| = \infty$. We call a vertex a *singular vertex* if it is either a sink or an infinite emitter.

For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ we let $\mathcal{G}^0$ denote the smallest subcollection of $\mathcal{P}(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections and finite unions. The following lemma gives us another description of $\mathcal{G}^0$.

Lemma 2.2 ([23], Lemma 2.12)**.** *If $\mathcal{G} := (G^0, \mathcal{G}^1, r, s)$ is an ultragraph, then*

$$\mathcal{G}^0 = \{ \bigcap_{e \in X_1} r(e) \cup \ldots \cup \bigcap_{e \in X_n} r(e) \cup F : X_1, \ldots, X_n \text{ are finite subsets of } \mathcal{G}^1$$
$$\text{and } F \text{ is a finite subset of } G^0 \}.$$

Furthermore, F may be chosen to be disjoint from $\bigcap_{e \in X_1} r(e) \cup \ldots \cup \bigcap_{e \in X_n} r(e)$.

Definition 2.3. If $\mathcal{G}$ is an ultragraph, a *Cuntz-Krieger $\mathcal{G}$-family* is a collection of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a collection of projections $\{p_A : A \in \mathcal{G}^0\}$ that satisfy

1. $p_\emptyset = 0$, $p_A p_B = p_{A \cap B}$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ for all $A, B \in \mathcal{G}^0$
2. $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$
3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

When A is a singleton set $\{v\}$, we shall write p_v in place of $p_{\{v\}}$.

Definition 2.4. If $\mathcal{G}$ is an ultragraph, we let $C^*(\mathcal{G})$ denote the C^*-algebra generated by a universal Cuntz-Krieger $\mathcal{G}$-family. It is proven in [23, Theorem 2.11] that $C^*(\mathcal{G})$ exists.

For $n \geq 2$ we define $\mathcal{G}^n := \{\alpha = \alpha_1 \ldots \alpha_n : \alpha_i \in \mathcal{G}^1 \text{ and } s(\alpha_{i+1}) \in r(\alpha_i)\}$ and $\mathcal{G}^* := \bigcup_{n=0}^\infty \mathcal{G}^n$. The map r extends naturally to $\mathcal{G}^*$, and we say that α has length $|\alpha| = n$ when $\alpha \in \mathcal{G}^n$. Note that the paths of length zero are the elements of $\mathcal{G}^0$, and when $A \in \mathcal{G}^0$ we define $s(A) = r(A) = A$.

If $\mathcal{G}$ is an ultragraph, then a *loop* is a path $\alpha \in \mathcal{G}^*$ with $|\alpha| \geq 1$ and $s(\alpha) \in r(\alpha)$. An *exit* for a loop is one of the following:

1. an edge $e \in \mathcal{G}^1$ such that there exists an i for which $s(e) \in r(\alpha_i)$ but $e \neq \alpha_{i+1}$
2. a sink w such that $w \in r(\alpha_i)$ for some i.

Condition (L): Every loop in $\mathcal{G}$ has an exit; that is, for any loop $\alpha := \alpha_1 \dots \alpha_n$ there is either an edge $e \in \mathcal{G}^1$ such that $s(e) \in r(\alpha_i)$ and $e \neq \alpha_{i+1}$ for some i, or there is a sink w with $w \in r(\alpha_i)$ for some i.

We mention that versions of the Cuntz-Krieger uniqueness theorem and the gauge-invariant uniqueness theorem have been proven for ultragraph algebras [23, Theorem 6.7 and Theorem 6.8].

Definition 2.5. If I is a countable set and A is an $I \times I$ matrix with entries in $\{0,1\}$, then we may form the ultragraph $\mathcal{G}_A := (G_A^0, \mathcal{G}_A^1, r, s)$ defined by $G_A^0 := \{v_i : i \in I\}$, $\mathcal{G}_A^1 := I$, $s(i) = v_i$ for all $i \in I$, and $r(i) = \{v_j : A_{\mathcal{G}}(i,j) = 1\}$.

Note that the edge matrix of $\mathcal{G}_A$ is A. If A is a countable $\{0,1\}$-matrix, then it was shown in [23, Theorem 4.5] that the Exel-Laca algebra $\mathcal{O}_A$ is canonically isomorphic to $C^*(\mathcal{G})$.

In [7] Exel and Laca associated a graph $\mathrm{Gr}(A)$ to A whose vertex matrix is equal to A. Specifically, one defines the vertices of $\mathrm{Gr}(A)$ to be I, and for each pair of vertices $i, j \in I$ one defines there to be $A(i,j)$ edges from i to j. We shall see that the ultragraph $\mathcal{G}_A$ can often tell us more about the structure of $\mathcal{O}_A \cong C^*(\mathcal{G})$ than the graph $\mathrm{Gr}(A)$ can.

3. Simplicity of Ultragraph Algebras

In [1, §4] the ideals of graph algebras were studied using saturated hereditary subsets of G^0. Our methods in this section will be similar, except that we now use saturated hereditary subcollections of $\mathcal{G}^0$. Although we could call these subsets of $\mathcal{G}^0$, we will refer to them as subcollections to emphasize that their elements are themselves subsets of G^0.

Definition 3.1. A subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *hereditary* if

(1) whenever e is an edge with $\{s(e)\} \in \mathcal{H}$, then $r(e) \in \mathcal{H}$
(2) $A, B \in \mathcal{H}$, implies $A \cup B \in \mathcal{H}$
(3) $A \in \mathcal{H}$, $B \in \mathcal{G}^0$, and $B \subseteq A$, imply that $B \in \mathcal{H}$.

Definition 3.2. A hereditary subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *saturated* if for any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$ we have that

$$\{r(e) : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq \mathcal{H} \quad \text{implies} \quad \{v\} \in \mathcal{H}.$$

The *saturation* of a hereditary collection $\mathcal{H}$ is the smallest saturated subcollection $\overline{\mathcal{H}}$ of $\mathcal{G}^0$ containing $\mathcal{H}$; the saturation $\overline{\mathcal{H}}$ is itself hereditary.

Remark 3.3. Note that if $\mathcal{H} \subseteq \mathcal{G}^0$ is a hereditary subcollection with $\{v\} \in \mathcal{H}$ for all $v \in G^0$, then $\mathcal{H} = \mathcal{G}^0$. This is because having $\mathcal{H}$ hereditary implies that $\mathcal{H}$ contains $r(e)$ for all $e \in \mathcal{G}^1$, and since $\mathcal{H}$ is closed under finite unions and intersections Lemma 2.2 then implies $\mathcal{H} = \mathcal{G}^0$.

Lemma 3.4. *Let $\mathcal{G}$ be an ultragraph and let I be an ideal in $C^*(\mathcal{G})$. Then $\mathcal{H} := \{A \in \mathcal{G}^0 : p_A \in I\}$ is a saturated hereditary subcollection of $\mathcal{G}^0$.*

Proof. Suppose $\{s(e)\} \in \mathcal{H}$. Then

$$p_{s(e)} \in I \Longrightarrow s_e = p_{s(e)} s_e \in I \Longrightarrow p_{r(e)} = s_e^* s_e \in I \Longrightarrow r(e) \in \mathcal{H}.$$

Also, if $A, B \in \mathcal{H}$, then

$$p_A, p_B \in I \Longrightarrow p_{A\cup B} = p_A + p_B - p_A p_B \in I \Longrightarrow A \cup B \in \mathcal{H}.$$

Finally, if $A \in H$, $B \in \mathcal{G}^0$, and $B \subseteq A$, then

$$p_A \in I \Longrightarrow p_B = p_B p_A \in I \Longrightarrow B \in H$$

so H is hereditary.

Furthermore, if $0 < |s^{-1}(v)| < \infty$ and $\{r(e) : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq H$, then $\{s_e : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq I$ and $p_v = \sum_{s(e)=v} s_e s_e^* \in I$ which implies that $\{v\} \in \mathcal{H}$. Thus H is saturated. □

For a hereditary subcollection $\mathcal{H} \subseteq \mathcal{G}^0$ let $I_\mathcal{H}$ denote the ideal in $C^*(\mathcal{G})$ generated by $\{p_A : A \in \mathcal{H}\}$.

Lemma 3.5. *Let $\mathcal{G}$ be an ultragraph and let $\mathcal{H}$ be a hereditary subcollection of $\mathcal{G}^0$. Then*

$$I_\mathcal{H} = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^* \text{ and } A \in \overline{\mathcal{H}}\}.$$

In particular, $I_\mathcal{H} = I_{\overline{\mathcal{H}}}$ and $I_\mathcal{H}$ is gauge invariant.

Proof. Note that $\{A \in \mathcal{G}^0 : p_A \in I_H\}$ is a saturated set containing $\mathcal{H}$ and therefore contains $\overline{\mathcal{H}}$. Thus $J := \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^* \text{ and } A \in \overline{\mathcal{H}}\}$ is contained in $I_\mathcal{H}$. For inclusion in the other direction, notice that any nonzero product of the form $s_\alpha p_A s_\beta^* s_\gamma p_B s_\delta^*$ collapses to another of the form $s_\mu p_C s_\nu^*$ and from an examination of the various possibilities and the hereditary property of $\overline{\mathcal{H}}$ we deduce that J is an ideal. Since J contains the generators of $I_\mathcal{H}$, it follows that $J := I_\mathcal{H}$. The last two remarks follow easily. □

Lemma 3.6. *Let $\mathcal{G}$ be an ultragraph for which $C^*(\mathcal{G})$ is simple. If $\mathcal{H}$ is a saturated hereditary subcollection of $\mathcal{G}^0$ and $K := \{v \in G^0 : \{v\} \in \mathcal{H}\}$, then for any $e \in \mathcal{G}^1$ we have that $r(e) \subseteq K$ implies that $r(e) \in \mathcal{H}$.*

Proof. If $\mathcal{H}$ is empty the claim holds vacuously. If $\mathcal{H} \neq \emptyset$, then since $C^*(\mathcal{G})$ is simple we know that $I_\mathcal{H} = C^*(\mathcal{G})$ and thus $p_{r(e)} \in I_\mathcal{H}$. By Lemma 3.5 there exist $\lambda_k \in \mathbb{C}$, $\alpha_k, \beta_k \in \mathcal{G}^*$, and $A_k, B_k \in \mathcal{H}$ for $1 \leq k \leq n$ such that

$$\|p_{r(e)} - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\| < 1.$$

Furthermore, since

$$\|p_{r(e)}\Big(p_{r(e)} - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\Big)\| \leq \|p_{r(e)} - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\|$$

we may assume that $s(\alpha_k) \in r(e)$ when $|\alpha_k| \geq 1$ and $s(\alpha_k) \subseteq r(e)$ when $|\alpha_k| = 0$. (We remind the reader that if $|\alpha| = 0$, then $\alpha = A$ for some $A \in \mathcal{G}^0$ and $s(\alpha) := A$.)

Now define $B := \bigcup_{k=1}^n s(\alpha_k)$. Since $B \subseteq r(e)$ we see that $q := p_{r(e)} - p_B$ is a projection. Furthermore,

$$\|q\| = \|q\Big(p_{r(e)} - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\Big)\| \leq \|p_{r(e)} - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\| < 1.$$

and since q is a projection this implies that $q = 0$. Therefore $p_{r(e)} = p_B$ and $r(e) = B = \bigcup_{k=1}^n s(\alpha_k) \in \mathcal{H}$. □

Lemma 3.7. *Let $\mathcal{G}$ be an ultragraph for which $C^*(\mathcal{G})$ is simple. If $\mathcal{H}$ is a saturated hereditary subcollection of $\mathcal{G}^0$, then either $\mathcal{H} = \mathcal{G}^0$ or $\mathcal{H} = \emptyset$.*

Proof. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$. Set $K := \{w \in G^0 : \{w\} \in \mathcal{H}\}$ and $S := G^0 \backslash K$. We define an ultragraph $\mathcal{F} = (F^0, \mathcal{F}^1, r_{\mathcal{F}}, s_{\mathcal{F}})$ as follows:

$$F^0 := S \qquad\qquad s_{\mathcal{F}}(e) := s(e)$$
$$\mathcal{F}^1 := \{e \in \mathcal{G}^1 : r(e) \cap S \neq \emptyset\} \qquad\qquad r_{\mathcal{F}}(e) := r(e) \cap S$$

Note that if $e \in \mathcal{F}^1$, then $r(e) \cap S \neq \emptyset$ so $r(e) \notin \mathcal{H}$ and since $\mathcal{H}$ is hereditary it follows that $\{s(e)\} \notin \mathcal{H}$ and $s(e) \in S$. Thus $s_{\mathcal{F}}$ is well-defined.

Let $\{s_e, p_A\}$ be the canonical Cuntz-Krieger $\mathcal{F}$-family in $C^*(\mathcal{F})$. For each $e \in \mathcal{G}^1$ and $A \in \mathcal{G}^0$ define

$$t_e := \begin{cases} s_e & \text{if } e \in \mathcal{F}^1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad q_A := p_{A \cap S}.$$

Note that if $A \in \mathcal{G}^0$, then by Lemma 2.2

$$A = \bigcap_{e \in X_1} r(e) \cup \ldots \cup \bigcap_{e \in X_n} r(e) \cup F$$

for some finite subsets $X_1, \ldots, X_n \subseteq \mathcal{G}^1$ and some finite subset $F \subseteq G^0$. Thus if

$$Y_i := \begin{cases} X_i & \text{if } X_i \subseteq \mathcal{F}^1 \\ \emptyset & \text{otherwise} \end{cases}$$

then we see that

$$\begin{aligned} A \cap S &= \bigcap_{e \in X_1} (r(e) \cap S) \cup \ldots \cup \bigcap_{e \in X_n} (r(e) \cap S) \cup (F \cap S) \\ &= \bigcap_{e \in Y_1} r_{\mathcal{F}}(e) \cup \ldots \cup \bigcap_{e \in Y_n} r_{\mathcal{F}}(e) \cup (F \cap S) \end{aligned}$$

which is in $\mathcal{F}^0$. Hence q_A is well-defined.

We shall now show that $\{t_e, q_A\}$ is a Cuntz-Krieger $\mathcal{G}$-family. Clearly, the t_e's have mutually orthogonal ranges since the s_e's do. Thus we simply need to verify the four properties of Definition 2.3.

(1) We have that $q_\emptyset = p_\emptyset = 0$, $q_A q_B = p_{A \cap S} p_{B \cap S} = p_{(A \cap B) \cap S} = q_{A \cap B}$, and $q_{A \cup B} = p_{(A \cup B) \cap S} = p_{(A \cap S) \cup (B \cap S)} = p_{(A \cap S)} + p_{(B \cap S)} - p_{(A \cap S) \cap (B \cap S)} = q_A + q_B - p_{(A \cap B) \cap S} = q_A + q_B - q_{A \cap B}$.

(2) If $e \in \mathcal{F}^1$, then $t_e^* t_e = s_e^* s_e = p_{r_{\mathcal{F}}(e)} = p_{r(e) \cap S} = q_{r(e)}$. On the other hand, if $e \notin \mathcal{F}^1$, then $r(e) \cap S = \emptyset$ so $q_{r(e)} = 0 = t_e^* t_e$.

(3) If $e \in \mathcal{F}^1$, then $s(e) \in S$ so $t_e t_e^* = s_e s_e^* \leq p_{s_{\mathcal{F}}(e)} = q_{s(e)}$. On the other hand, if $e \notin \mathcal{F}^1$, then $t_e t_e^* = 0 \leq q_{s(e)}$.

(4) Let $v \in G^0$ and $0 < |s^{-1}(v)| < \infty$. If $v \notin S$, then $\{v\} \in \mathcal{H}$ and $r(e) \in \mathcal{H}$ so $r(e) \cap S = \emptyset$ and

$$\sum_{\{e \in \mathcal{G}^1 : s(e) = v\}} t_e t_e^* = 0 = q_{r(e)}.$$

If $v \in S$, then since $s_{\mathcal{F}}^{-1}(v) \subseteq s^{-1}(v)$ we have that $|s_{\mathcal{F}}^{-1}(v)| < \infty$. Also note that $r(e) \cap S = \emptyset$ implies $r(e) \in \mathcal{H}$ by Lemma 3.6. Thus the fact that

$\{v\} \notin \mathcal{H}$ and the fact that $\mathcal{H}$ is saturated imply that there is at least one edge e with $r(e) \cap S \neq \emptyset$. Hence $0 < |s_{\mathcal{F}}^{-1}(v)|$. Thus

$$\sum_{\{e \in \mathcal{G}^1 : s(e)=v\}} t_e t_e^* = \sum_{\{e \in \mathcal{F}^1 : s(e)=v\}} t_e t_e^* + \sum_{\{e \in (\mathcal{G}^1 \setminus \mathcal{F}^1) : s(e)=v\}} t_e t_e^*$$
$$= \sum_{\{e \in \mathcal{F}^1 : s_{\mathcal{F}}(e)=v\}} s_e s_e^* + 0 = p_{s_{\mathcal{F}}(e)} = q_{s(e)}.$$

Now since $\{t_e, q_v\}$ is a Cuntz-Krieger $\mathcal{G}$-family with $q_v = 0$ if and only if $v \notin S$, the universal property gives a homomorphism $\phi : C^*(\mathcal{G}) \to C^*(\mathcal{F})$ whose kernel contains only those projections corresponding to vertices that are not in S. Since $C^*(\mathcal{G})$ is simple, the kernel of ϕ is either $C^*(\mathcal{G})$ or $\{0\}$. Thus S is either $\emptyset$ or G^0, and K is either G^0 or $\emptyset$. Since $\mathcal{H}$ is a saturated hereditary subset, this implies that either $\mathcal{H} = \emptyset$ or $\mathcal{H} = \mathcal{G}^0$. □

The following proof is modeled after that of [1, Theorem 4.1(c)].

Lemma 3.8. *Let $\mathcal{G}$ be an ultragraph. If $\mathcal{G}$ has a loop with no exits, then $C^*(\mathcal{G})$ contains an ideal Morita equivalent to $C(\mathbb{T})$.*

Proof. Let $C^*(\mathcal{G}) = C^*(\{s_e, p_A\})$ and $\alpha = \alpha_1 \dots \alpha_n$ be a loop in $\mathcal{G}$ with no exits. This implies that $r(\alpha_i) = \{s(\alpha_{i+1})\}$ for $1 \leq i < n$, and $r(\alpha_n) = \{s(\alpha_1)\}$. In particular, the α_i's have ranges that are singleton sets. Define $X := \{s(\alpha_i)\}_{i=1}^n$ and $q_X := \sum_{v \in X} p_v$. If $\mathcal{H}$ equals the (finite) collection of all subsets of X, then $\mathcal{H}$ is a hereditary subset of $\mathcal{G}^0$. We shall show that $I_{\mathcal{H}} = I_{\overline{\mathcal{H}}}$ is Morita equivalent to $C(\mathbb{T})$.

Define $G^1 := \{\alpha_i\}_{i=1}^n$ and let G be the graph $G := (X, G^1, r, s)$. We claim that $q_X I_{\overline{\mathcal{H}}} q_X$ is generated by the Cuntz-Krieger G-family $\{s_e, p_v : e \in G^1, v \in X\}$. Certainly this family lies in the corner. On the other hand, if $\alpha, \beta \in \mathcal{G}^*$ and $A \in \mathcal{G}^0$, then $q_X s_\alpha p_A s_\beta^* q_X = 0$ unless both α and β have sources in X. Thus the claim is verified and the gauge-invariant uniqueness theorem for C^*-algebras of graphs [1, Theorem 2.1] implies that $q_X I_{\overline{\mathcal{H}}} q_X \cong C(G)$. To see that this is a full corner of $I_{\mathcal{H}}$, suppose that J is an ideal in $I_{\mathcal{H}}$ containing $q_X I_{\overline{\mathcal{H}}} q_X$. Then J is an ideal of $C^*(\mathcal{G})$ and Lemma 3.4 implies that $\{A \in \mathcal{G}^0 : p_A \in J\}$ is a saturated hereditary subcollection containing $\{\{v\} : v \in X\}$ and hence containing $\mathcal{H}$. But this implies that J contains the generators of $I_{\overline{\mathcal{H}}}$ and hence is all of $I_{\overline{\mathcal{H}}}$.

Therefore, $I_{\overline{\mathcal{H}}}$ is Morita equivalent to $C^*(G)$. Since G is a loop of length n, we see from [15, Theorem 2.4] that $C^*(G) \cong C(\mathbb{T}) \otimes M_n(\mathbb{C})$ which is Morita equivalent to $C(\mathbb{T})$. □

Lemma 3.9. *Let $\mathcal{G}$ be an ultragraph such that $C^*(\mathcal{G})$ is simple. Then every loop in $\mathcal{G}$ has an exit.*

Proof. If $\mathcal{G}$ contained a loop with no exits, then Lemma 3.8 would imply that $C^*(\mathcal{G})$ contains an ideal Morita equivalent to $C(\mathbb{T})$. Hence $C^*(\mathcal{G})$ could not be simple. □

The following is a generalization of [22, Theorem 8]

Theorem 3.10. *If $\mathcal{G}$ is an ultragraph, then $C^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ satisfies:*

1. *every loop in $\mathcal{G}$ has an exit*
2. *the only saturated hereditary subcollections of $\mathcal{G}^0$ are $\mathcal{G}^0$ and $\emptyset$.*

Proof. Suppose that $C^*(\mathcal{G})$ is simple. Then Lemma 3.9 implies that every loop in $\mathcal{G}$ must have an exit. Furthermore, if $\mathcal{H}$ is a saturated hereditary subcollection of $\mathcal{G}^0$, then it follows from Lemma 3.7 that either $\mathcal{H} = \mathcal{G}^0$ or $\mathcal{H} = \emptyset$.

Conversely, suppose that $\mathcal{G}$ satisfies the two properties above. If I is an ideal in $C^*(\mathcal{G})$, then Lemma 3.4 tells us that $\mathcal{H} := \{A \in \mathcal{G}^0 : p_A \in I\}$ is a saturated hereditary subcollection of $\mathcal{G}^0$. Hence $\mathcal{H}$ equals either $\mathcal{G}^0$ or $\emptyset$. If $\mathcal{H} = \mathcal{G}^0$, then clearly $I = C^*(\mathcal{G})$. On the other hand, if $\mathcal{H} = \emptyset$, then since every loop in $\mathcal{G}$ has an exit we may use the Cuntz-Krieger Uniqueness Theorem to conclude that the projection $\pi : C^*(\mathcal{G}) \to C^*(\mathcal{G})/I$ is injective, and thus $I = \{0\}$. □

Recall from [4, Corollary 2.15] and [17, Theorem 4] that if G is a graph, then $C^*(G)$ is simple if and only if every loop in G has an exit, G is cofinal, and $G^0 \geq \{v\}$ for every singular vertex $v \in G^0$. We shall use the previous theorem to obtain an analogous characterization for ultragraph algebras. Recall that infinite emitters in a graph correspond to infinite sets of the form $r(e) \in \mathcal{G}^0$. In fact, if G is a graph with vertex matrix A, then in the ultragraph $\mathcal{G}_A$ the set $r(e)$ is finite for all $e \in \mathcal{G}_A^1$ if and only if G has no infinite emitters.

We first extend the notions of $\geq$ and cofinality to ultragraphs. If $\mathcal{G}$ is an ultragraph and $v, w \in G^0$, we write $w \geq v$ to mean that there exists a path $\alpha \in \mathcal{G}^*$ with $s(\alpha) = w$ and $v \in r(\alpha)$. Also, we write $G^0 \geq \{v\}$ to mean that $w \geq v$ for all $w \in G^0$. We say that $\mathcal{G}$ is *cofinal* if for every infinite path $\alpha := e_1 e_2 \ldots$ and every vertex $v \in G^0$ there exists an $i \in \mathbb{N}$ such that $v \geq s(e_i)$.

In addition, we need a new notion of reachability. If $v \in G^0$ and $A \subseteq G^0$, then we write $v \to A$ to mean that there exist a finite number of paths $\alpha_1, \ldots \alpha_n \in \mathcal{G}^*$ such that $s(\alpha_i) = v$ for all $1 \leq i \leq n$ and $A \subseteq \bigcup_{i=1}^n r(\alpha_i)$. Note that if $A = \{w\}$, then $v \to \{w\}$ if and only if $v \geq w$.

Theorem 3.11. *If $\mathcal{G}$ is an ultragraph, then $C^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ satisfies:*

1. *every loop in $\mathcal{G}$ has an exit*
2. *$\mathcal{G}$ is cofinal*
3. *$G^0 \geq \{v\}$ for every singular vertex $v \in G^0$*
4. *If $e \in \mathcal{G}^1$ is an edge for which the set $r(e)$ is infinite, then for every $w \in G^0$ there exists a set $A_w \subseteq r(e)$ for which $r(e) \backslash A_w$ is finite and $v \to A_w$.*

In order to prove this result we need a lemma.

Lemma 3.12. *Let $\mathcal{G}$ be an ultragraph and let $\mathcal{H} \subseteq \mathcal{G}^0$ be a hereditary subset. Set $\mathcal{H}_0 := \mathcal{H}$ and for $n \in \mathbb{N}$ define*

$$\mathcal{H}_{n+1} := \{A \cup F : A \in \mathcal{H}_n \text{ and } F \text{ is a finite subset of } S_n\}$$

where $S_n := \{w \in G^0 : 0 < |s^{-1}(w)| < \infty \text{ and } \{r(e) : s(e) = w\} \subseteq \mathcal{H}_n\}$. Then

$$\overline{\mathcal{H}} = \bigcup_{i=0}^{\infty} \mathcal{H}_i$$

and every $X \in \overline{\mathcal{H}}$ has the form $X = A \cup F$ for some $A \in \mathcal{H}$ and some finite set $F \subseteq \bigcup_{i=1}^{\infty} S_i$.

Proof. To see that $\bigcup_{i=0}^{\infty} \mathcal{H}_i \subseteq \overline{\mathcal{H}}$, first note that $\mathcal{H}_0 \subseteq \overline{\mathcal{H}}$. Also whenever $\mathcal{H}_n \subseteq \overline{\mathcal{H}}$, then because $\overline{\mathcal{H}}$ is saturated we have that $F \in \overline{\mathcal{H}}$ for any finite subset $F \subseteq S_n$, and hence $\mathcal{H}_{n+1} \subset \overline{\mathcal{H}}$. Thus by induction we have $\mathcal{H}_n \subseteq \overline{\mathcal{H}}$ for all n.

To see that $\overline{\mathcal{H}} \subseteq \bigcup_{i=0}^{\infty} \mathcal{H}_i$ we shall show that $\bigcup_{i=0}^{\infty} \mathcal{H}_i$ is a saturated hereditary subcollection. We shall begin by proving inductively that each $\mathcal{H}_i$ is hereditary. For the base case, we have by hypothesis that $\mathcal{H}_0 := \mathcal{H}$ is hereditary. Now assume that $\mathcal{H}_n$ is hereditary and consider $\mathcal{H}_{n+1}$. If $\{s(e)\} \in \mathcal{H}_{n+1}$, then by the definition of $\mathcal{H}_{n+1}$ either $\{s(e)\} \in \mathcal{H}_n$ or $s(e) \in S_n$. In either case, $r(e) \in \mathcal{H}_n \subseteq \mathcal{H}_{n+1}$. To see that $\mathcal{H}_{n+1}$ is closed under subsets, Let $A_1 \cup F_1$ and $A_2 \cup F_2$ be typical elements of $\mathcal{H}_{n+1}$. Then $(A_1 \cup F_1) \cup (A_2 \cup F_2) = (A_1 \cup A_2) \cup (F_1 \cup F_2)$ which is in $\mathcal{H}_{n+1}$ because $\mathcal{H}_n$ is closed under unions. Finally, suppose that $A \cup F$ is a typical element of $\mathcal{H}_{n+1}$ and that $B \in \mathcal{G}^0$ with $B \subseteq A \cup F$. Then $A \cap B \subseteq A$ and since $\mathcal{H}_n$ is hereditary $A \cap B \in \mathcal{H}_n \subseteq \mathcal{H}_{n+1}$. Also, $B \cap F \subseteq F$ so $B \cap F$ is a finite subset of S_n. Thus $B = (B \cap A) \cup (B \cap F) \in \mathcal{H}_{n+1}$ and $\mathcal{H}_{n+1}$ is hereditary.

Since $\bigcup_{i=0}^{\infty} \mathcal{H}_i$ is the union of hereditary sets, it follows that $\bigcup_{i=0}^{\infty} \mathcal{H}_i$ itself is hereditary. To see that $\bigcup_{i=0}^{\infty} \mathcal{H}_i$ is also saturated, let $v \in G^0$ be a vertex with $0 < |s^{-1}(v)| < \infty$ and $\{r(e) : s(e) = v\} \subseteq \bigcup_{i=0}^{\infty} \mathcal{H}_i$. Since $\mathcal{H}_i \subseteq \mathcal{H}_{i+1}$ and since there are only finitely many edges with source v, we see that there exists $n \in \mathbb{N}$ such that $\{r(e) : s(e) = v\} \subseteq \mathcal{H}_n$. Thus $v \in S_n$ and $\{v\} \in \mathcal{H}_{n+1} \subseteq \bigcup_{i=0}^{\infty} \mathcal{H}_i$. Hence $\bigcup_{i=0}^{\infty} \mathcal{H}_i$ is saturated.

Therefore $\overline{\mathcal{H}} = \bigcup_{i=0}^{\infty} \mathcal{H}_i$ and to prove the claim, we let $X \in \bigcup_{i=0}^{\infty} \mathcal{H}_i$. Then $X \in \mathcal{H}_n$ for some $n \in \mathbb{N}$ and $X = A_{n-1} \cup F_{n-1}$ for some $A_{n-1} \in \mathcal{H}_{n-1}$ and some finite subset $F_{n-1} \subseteq S_{n-1}$. Similarly, $A_{n-1} = A_{n-2} \cup F_{n-2}$ for some $A_{n-2} \in \mathcal{H}_{n-2}$ and some finite subset $F_{n-2} \subseteq S_{n-2}$. Continuing inductively we see that $A = A_0 \cup (F_{n-1} \cup \ldots \cup F_1)$ where the F_i's are all finite sets. □

Proof of Necessity in Theorem 3.11. Suppose that $C^*(\mathcal{G})$ is simple. By Theorem 3.10 we see that every loop in $\mathcal{G}$ has an exit and the only saturated hereditary subsets of $\mathcal{G}^0$ are $\mathcal{G}^0$ and $\emptyset$.

Let $\alpha = e_1 e_2 \ldots$ be an infinite path and set $K := \{w \in G^0 : w \not\geq s(e_i) \text{ for all } i\}$. Also define $\mathcal{H} := \{A \in \mathcal{G}^0 : A \subseteq K\}$. Then one can verify that $\mathcal{H}$ is a saturated hereditary subcollection of $\mathcal{G}^0$. Since $\{s(e_1)\} \notin \mathcal{H}$ we see that $\mathcal{H}$ is not all of $\mathcal{G}^0$. Thus $\mathcal{H} = \emptyset$ and $\mathcal{G}$ is cofinal.

Let $v \in G^0$ be a singular vertex. Fix any vertex $w \in G^0$ and define $K := \{x \in G^0 : w \geq x\}$. Also let $\mathcal{H} := \{A \in \mathcal{G}^0 : A \subseteq K\}$. Then $\mathcal{H}$ is a hereditary subcollection of $\mathcal{G}^0$. If $\overline{\mathcal{H}}$ is the saturation of $\mathcal{H}$, then $\overline{\mathcal{H}}$ is nonempty because $\{w\} \in \mathcal{H}$. Hence $\overline{\mathcal{H}} = \mathcal{G}^0$. Now, using the notation of Lemma 3.12, we see that $v \notin S_i$ for all i because v is a singular vertex. Therefore it follows from Lemma 3.12 that $\{v\} \in \overline{\mathcal{H}}$ implies that $\{v\} \in \mathcal{H}$. Thus $v \in K$ and $w \geq v$. Hence $G^0 \geq \{v\}$.

Let $e \in \mathcal{G}^1$ be an edge such that $r(e)$ is an infinite set. Fix $w \in G^0$ and set $\mathcal{H} := \{A \in \mathcal{G}^0 : w \to A\}$. To see that $\mathcal{H}$ is hereditary suppose that $\{s(f)\} \in \mathcal{H}$. Then $w \to \{s(f)\}$ and hence $v \geq s(f)$. Thus there exists a path β with $s(\beta) = w$ and $s(f) \in r(\beta)$. But then we see that $w \to r(f)$ via the path βf. Additionally, it is easy to see that $\mathcal{H}$ is closed under unions and subsets. Since $\{w\} \in \mathcal{H}$, it follows that $\mathcal{H}$ is nonempty, and hence $\overline{\mathcal{H}} = \mathcal{G}^0$. Thus $r(e) \in \overline{\mathcal{H}}$. By Lemma 3.12 it follows that $r(e) = A_w \cup F$ for some $A_w \in \mathcal{H}$ and some finite set F. But then $w \to A_w$ and $r(e) \backslash A_w$ is finite. □

Proof of Sufficiency in Theorem 3.11. Suppose that $\mathcal{G}$ satisfies the four conditions stated in Theorem 3.11. In light of Theorem 3.10 it suffices to show that the only saturated hereditary subcollections of $\mathcal{G}^0$ are $\emptyset$ and $\mathcal{G}^0$.

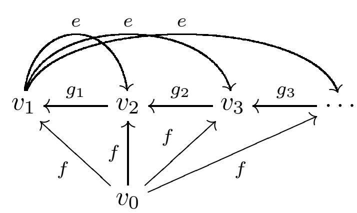
Let $\mathcal{H}$ be a nonempty saturated hereditary subcollection of $\mathcal{G}^0$. We shall show that for every $w \in G^0$ with $\{w\} \notin \mathcal{H}$ there exists an edge $e \in \mathcal{G}^1$ such that $s(e) = w$ and $r(e)$ contains a vertex w' for which $\{w'\} \notin \mathcal{H}$.

If $\{w\} \notin \mathcal{H}$, then since $G^0 \geq \{v\}$ for every singular vertex v it follows that w is not a singular vertex. Therefore, since $\mathcal{H}$ is saturated, there exists an edge e such that $s(e) = w$ and $r(e) \notin \mathcal{H}$. If $r(e)$ is finite, then because $\mathcal{H}$ is closed under unions, there must exist a vertex $w' \in r(e)$ such that $\{w'\} \notin \mathcal{H}$. If $r(e)$ is infinite, then choose some $x \in G^0$ for which $\{x\} \in \mathcal{H}$. Then there exists $A_x \subseteq r(e)$ such that $w \to A_x$ and $r(e) \backslash A_x$ is a finite set. Let $\alpha_1, \ldots, \alpha_n$ be paths with $s(\alpha_i) = x$ and $A_x \subseteq \bigcup_{i=1}^n r(\alpha_i)$. Since $\mathcal{H}$ is hereditary, it follows that $\bigcup_{i=1}^n r(\alpha_i) \in \mathcal{H}$. Now we must have that one of the vertices in $r(e) \backslash A_x$ is not in $\mathcal{H}$. For otherwise, $r(e) \backslash A_x \in \mathcal{H}$ and $\bigcup_{i=1}^n r(\alpha_i) \cup (r(e) \backslash A_x)$ is an element in $\mathcal{H}$ containing $r(e)$ which contradicts the fact that $r(e) \notin \mathcal{H}$.

Now suppose that there exists $w_1 \in G^0$ such that $\{w_1\} \notin \mathcal{H}$. From the argument in the preceding paragraph there exists an edge e_1 and a vertex w_2 such that $s(e_1) = w_1$, $w_2 \in r(e_1)$, and $\{w_2\} \notin \mathcal{H}$. Continuing inductively, we create an infinite path $e_1 e_2 e_3 \ldots$ with $\{s(e_i)\} \notin \mathcal{H}$ for all i. But this contradicts the cofinality of $\mathcal{G}$. Hence $\mathcal{H}$ must be all of $\mathcal{G}^0$. □

Remark 3.13. Note that if $\mathcal{G}$ is an ultragraph with two sinks v_1 and v_2, then $v_1 \ngeq v_2$ and hence $G^0 \ngeq v_2$. Therefore if $\mathcal{G}$ is an ultragraph with two or more sinks, then $C^*(\mathcal{G})$ is not simple. In addition, if $\mathcal{G}$ has exactly one sink and $C^*(\mathcal{G})$ is simple, then $\mathcal{G}$ contains no infinite paths because the sink is unable to reach any infinite path.

Example 3.14. Let $\mathcal{G}$ be the ultragraph

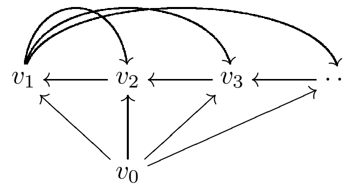


Since any loop in $\mathcal{G}$ must contain the edge e, we see that every loop has an exit. In addition, let $\mathcal{H}$ be a saturated hereditary subcollection of $\mathcal{G}^0$. If $\mathcal{H}$ is nonempty, then there is some singleton set $\{v\} \in \mathcal{H}$ for $v \in G^0$. Because $\mathcal{H}$ is hereditary and because each v_i can be reached from any other vertex, we see that $\mathcal{H}$ must contain $\{v_i\}$ for $1 \leq i < \infty$. Since $s(e) = v_1$ and $\{v_1\} \in \mathcal{H}$ it follows that $r(e) = \{v_2, v_3, v_4, \ldots\} \in \mathcal{H}$. Hence $r(f) = \{v_1, v_2, v_3, \ldots\} = \{v_1\} \cup r(e) \in \mathcal{H}$. Since $\mathcal{H}$ is saturated we also have that $\{v_0\} \in \mathcal{H}$. Thus $\mathcal{H}$ contains $r(e)$, $r(f)$, and $\{v\}$ for all $v \in G^0$. Consequently $\mathcal{H} = \mathcal{G}^0$. It follows from Theorem 3.10 that $C^*(\mathcal{G})$ is simple.

Example 3.15. Consider the infinite matrix

$$A = \begin{pmatrix} 1&1&1&1&1&1& \\ 0&0&1&1&1&1& \\ 0&1&0&0&0&0&\cdots \\ 0&0&1&0&0&0& \\ 0&0&0&1&0&0& \\ 0&0&0&0&1&0& \\ &&\vdots&&&&\ddots \end{pmatrix}$$

The graph $\mathrm{Gr}(A)$ associated to A is



Since v_0 is an infinite emitter, we see that $H := \{v_1, v_2, v_3, \ldots\}$ is a saturated hereditary subset of G^0 that gives rise to a nontrivial ideal. Therefore $C^*(\mathrm{Gr}(A))$ is not simple.

However, the ultragraph $\mathcal{G}_A$ associated to A is the ultragraph shown in Example 3.14, and as we saw there $C^*(\mathcal{G}_A)$ is simple. Note that $\mathcal{G}_A$ has no infinite emitters or sinks, and in fact, $|s^{-1}(v)| = 1$ for all $v \in G^0$. It then follows from [23, Theorem 4.5] that $C^*(\mathcal{G}_A) \cong \mathcal{O}_A$. Thus $\mathcal{O}_A$ is simple.

This example shows that the ultragraph $\mathcal{G}_A$ is a better tool for studying $\mathcal{O}_A$ than the graph $\mathrm{Gr}(A)$. As we saw, the fact that $\mathcal{O}_A$ is simple is reflected in the ultragraph $\mathcal{G}_A$, but it is difficult to see from the graph $\mathrm{Gr}(A)$. In addition, $C^*(\mathrm{Gr}(A))$ is very different from $\mathcal{O}_A$ whereas $C^*(\mathcal{G}_A) \cong \mathcal{O}_A$.

4. AF and purely infinite ultragraph algebras

Theorem 4.1. *Let $\mathcal{G}$ be an ultragraph. Then $C^*(\mathcal{G})$ is an AF-algebra if and only if $\mathcal{G}$ has no loops.*

Proof. Let $\mathcal{F}$ be a desingularization of $\mathcal{G}$ [23, Definition 6.3]. Then since the class of AF-algebras is closed under stable isomorphism [6, Theorem 9.4], and since $\mathcal{F}$ has loops if and only if $\mathcal{G}$ has loops, we see that it suffices to prove the claim for ultragraphs with no singular vertices.

Suppose $\mathcal{G}$ has no singular vertices. If $\mathcal{G}$ has no loops, then write $\mathcal{G}^1 := \bigcup_{n=1}^{\infty} F_n$ as the increasing union of finite subsets F_n, and let B_n be the C^*-subalgebra of $C^*(\mathcal{G})$ generated by $\{s_e : e \in F_n\}$. By [23, Corollary 5.4] there are isomorphisms $\phi_n : C^*(G_{F_n}) \to B_n$. Since $\mathcal{G}$ has no loops, it follows from [23, Lemma 5.6] that each G_{F_n} has no loops. Since G_{F_n} is a finite graph with no loops, $C^*(G_{F_n}) \cong B_n$ is a finite-dimensional C^*-algebra [15, Corollary 2.3]. Because $\mathcal{G}$ has no singular vertices, the s_e's are dense in $C^*(\mathcal{G})$ and $C^*(\mathcal{G}) = \overline{\bigcup_{n=1}^{\infty} B_n}$. Thus $C^*(\mathcal{G})$ is the direct limit of finite-dimensional C^*-algebras, and consequently $C^*(\mathcal{G})$ is an AF-algebra.

Conversely, suppose that $\mathcal{G}$ has a loop $\alpha := \alpha_1 \ldots \alpha_n$.

Case I: α has an exit.

Because $\mathcal{G}$ has no sinks, we may assume without loss of generality that there exists an edge $f \in \mathcal{G}^1$ with $f \neq \alpha_1$ and $s(f) \in r(\alpha_n)$. Now

$$p_{r(e)} = s_\alpha^* s_\alpha \sim s_\alpha s_\alpha^* \leq s_{\alpha_1} s_{\alpha_1}^* < s_{\alpha_1} s_{\alpha_1}^* + s_f s_f^* \leq p_{s(\alpha)} \leq p_{r(\alpha)}$$

and so $p_{r(\alpha)}$ is an infinite projection. Since a projection in an AF-algebra is equivalent to one in a finite-dimensional subalgebra it cannot be infinite. Hence $C^*(\mathcal{G})$ is not AF.

Case II: α has no exits.

Since $\mathcal{G}$ has no sinks, it follows from Lemma 3.8 that $C^*(\mathcal{G})$ contains an ideal Morita equivalent to $C(\mathbb{T})$ which is not AF. Hence $C^*(\mathcal{G})$ cannot be AF. $\square$

We say that a vertex w *connects to a loop* $\alpha := \alpha_1 \dots \alpha_n$ if there exists a path $\gamma \in \mathcal{G}^*$ with $s(\gamma) = w$ and $s(\alpha_i) \in r(\gamma)$ for some $1 \leq i \leq n$. Note that if w is a sink on a loop (i.e. $w \in r(\alpha_i)$ for some i), then w does not connect to a loop.

Lemma 4.2. *Let $\mathcal{G}$ be an ultragraph with no singular vertices and let A be the edge matrix of $\mathcal{G}$. If every vertex in $\mathcal{G}$ connects to a loop, then every vertex in* $\mathrm{Gr}(A)$ *connects to a loop.*

Proof. Let a be a vertex in $\mathrm{Gr}(A)$. Then $a \in \mathrm{Gr}(A)^0 = \mathcal{G}^1$. Choose any vertex $w \in G^0$ with $w \in r(a)$. By hypothesis, w connects to a loop $\alpha = \alpha_1 \dots \alpha_n$ in $\mathcal{G}$. Without loss of generality we may assume that there exists a path $\gamma = \gamma_1 \dots \gamma_m$ in $\mathcal{G}$ with $s(\gamma) = w$ and $s(\alpha_1) \in r(\gamma)$. Now since $A(\alpha_i, \alpha_{i+1}) = 1$ for $1 \leq i \leq n-1$ and $A(\alpha_n, \alpha_1) = 1$, we see that there exists a loop in $\mathrm{Gr}(A)$ with vertices $\alpha_1, \dots \alpha_n$. Furthermore, since $A(a, \gamma_1) = 1$, $A(\gamma_i, \gamma_{i+1}) = 1$ for $1 \leq i \leq m-1$, and $A(\gamma_m, \alpha_1) = 1$ we see that there is a path in $\mathrm{Gr}(A)$ from a to this loop. $\square$

In [2] Cuntz introduced the algebras $\mathcal{O}_n$ and proved that they were simple and had a property which he called "purely infinite". Since that time the property of being purely infinite has been reformulated in a number of ways for simple C^*-algebras, and this has caused some problems in deciding how to extend the notion to the non-simple case. In fact, various authors have used different definitions of purely infinite for non-simple C^*-algebras, and although these definitions agree in the simple case, they are not equivalent in general. In this paper we shall use the definition that was used in [15], [7], and [1]:

Definition 4.3. A C^*-algebra A is *purely infinite* if every nonzero hereditary subalgebra of A contains an infinite projection.

A competing definition is due to Kirchberg and Rørdam [13]: Every nonzero hereditary subalgebra of *every quotient* of A contains an infinite projection. Note that the definition that we use is weaker than this, but that both definitions agree in the simple case.

Theorem 4.4. *Let $\mathcal{G}$ be an ultragraph. Then $C^*(\mathcal{G})$ is purely infinite if and only if every loop in $\mathcal{G}$ has an exit and every vertex in $\mathcal{G}$ connects to a loop.*

Proof. If $\mathcal{G}$ contains a loop without an exit, then Lemma 3.8 tells us that $C^*(\mathcal{G})$ contains an ideal Morita equivalent to a commutative C^*-algebra. Since ideals are hereditary subalgebras this implies that $C^*(\mathcal{G})$ is not purely infinite.

Now suppose that every loop in $\mathcal{G}$ contains an exit, but that there is a vertex $v \in G^0$ that does not connect to a loop. Let $F^0 := \{w \in G^0 : v \geq w\}$ and $\mathcal{F}^1 := \{e \in \mathcal{G}^1 : s(e) \in F^0\}$. Note that $e \in \mathcal{F}^1$ implies $r(e) \subseteq F^0$, and thus r and s restrict in such a way that we may form the ultragraph $\mathcal{F} := (F^0, \mathcal{F}^1, r, s)$. Let $\{s_e, p_A\}$ be the generating Cuntz-Krieger $\mathcal{G}$-family. Then Lemma 2.2 implies that $\mathcal{F}^0 \subseteq \mathcal{G}^0$. This combined with the fact that $s(e) \in F^0$ implies $e \in \mathcal{F}^0$ shows that $\{s_e, p_A : e \in \mathcal{F}^1, A \in \mathcal{F}^0\} \subseteq C^*(\mathcal{G})$ is a Cuntz-Krieger $\mathcal{F}$-family. Since v does not connect to a loop, we see that $\mathcal{F}$ has no loops and hence $\mathcal{F}$ satisfies Condition (L). Thus by the Cuntz-Krieger uniqueness theorem [23, Theorem 6.7], we see that $C^*(\mathcal{F})$ is isomorphic to the subalgebra $B := \overline{\mathrm{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{F}^0, A \in \mathcal{F}^0\}$. We shall show that this subalgebra is hereditary. Let $\alpha, \beta, \gamma, \delta \in \mathcal{F}^*$ and $A, B \in \mathcal{F}^0$. Then for any $\mu, \nu \in \mathcal{G}^*$ and $C \in \mathcal{G}^0$ we see from a consideration of cases that $s_\alpha p_A s_\beta^* (s_\mu p_C s_\nu^*) s_\gamma p_B s_\delta^*$ will have the form $s_\epsilon p_D s_\sigma^*$ for some $\epsilon, \sigma \in \mathcal{F}^*$ and $D \in \mathcal{F}^0$.

Since these elements span dense subsets in $C^*(\mathcal{G})$ and B, we see that for all $b, b' \in B$ and $a \in C^*(\mathcal{G})$ we have $bab' \in B$. It follows from [16, Theorem 3.2.2] that B is hereditary. But now, since $\mathcal{F}$ does not contain any loops, Theorem 4.1 implies that $C^*(\mathcal{F}) \cong B$ is AF. Hence $C^*(\mathcal{G})$ cannot be purely infinite.

Conversely, suppose that $\mathcal{G}$ is an ultragraph in which every loop has an exit and every vertex connects to a loop. Let $\mathcal{F}$ be a desingularization of $\mathcal{G}$ [23, Definition 6.3]. Then $\mathcal{F}$ satisfies Condition (L) if and only if $\mathcal{G}$ does, and also every vertex in $\mathcal{F}$ connects to a loop if and only if every vertex in $\mathcal{G}$ connects to a loop. Since $C^*(\mathcal{G})$ is isomorphic to a full corner of $C^*(\mathcal{F})$ and because pure infiniteness is preserved by passing to corners, it therefore suffices to prove the converse for ultragraphs with no singular vertices.

Let us therefore assume that $\mathcal{G}$ has no singular vertices. If A is the edge matrix of $\mathcal{G}$, then it follows from [23, Theorem 4.5] that $\mathcal{O}_A \cong C^*(\mathcal{G})$. Now since $\mathcal{G}$ satisfies Condition (L), it follows from [23, Lemma 5.8] that $\mathrm{Gr}(A)$ satisfies Condition (L). Also, since every vertex in $\mathcal{G}$ connects to a loop, it follows from Lemma 4.2 that every vertex in $\mathrm{Gr}(A)$ connects to a loop. Therefore, [7, Theorem 16.2] implies that $\mathcal{O}_A \cong C^*(\mathcal{G})$ is purely infinite. □

Proposition 4.5 (The Dichotomy)**.** *Let $\mathcal{G}$ be an ultragraph for which $C^*(\mathcal{G})$ simple. Then*

(1) *$C^*(\mathcal{G})$ is AF if $\mathcal{G}$ has no loops.*
(2) *$C^*(\mathcal{G})$ is purely infinite if $\mathcal{G}$ contains a loop.*

Proof. Since $C^*(\mathcal{G})$ is simple, it follows from Theorem 3.11 that $\mathcal{G}$ is cofinal and satisfies Condition (L). If $\mathcal{G}$ has no loops, then $C^*(\mathcal{G})$ is AF by Theorem 4.1. If $\mathcal{G}$ has a loop, then every vertex connects to that loop due to cofinality, and $C^*(\mathcal{G})$ is purely infinite by Theorem 4.4. □

5. An ultragraph algebra that is neither an Exel-Laca algebra nor a graph algebra

It was shown in [23, Proposition 3.1] that graph algebras are ultragraph algebras and in [23, Theorem 4.5] that Exel-Laca algebras are ultragraph algebras. Here we show that this containment is strict. We provide an example of an ultragraph algebra that is neither an Exel-Laca algebra nor an ultragraph algebra.

Let A be the countably infinite matrix

$$A = \begin{pmatrix} 1&0&0&1&1&1&1&&\\ 0&1&0&1&1&1&1&&\\ 0&0&1&1&1&1&1&&\\ 1&0&0&1&0&0&0&\cdots&\\ 0&1&0&0&1&0&0&&\\ 0&0&1&0&0&1&0&&\\ 0&0&0&1&0&0&1&&\\ &&&\vdots&&&&\ddots& \end{pmatrix}.$$

Lemma 5.1. *If A is as above, then $K_0(\mathcal{O}_A) \cong 0$ and $K_1(\mathcal{O}_A) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Let I be the index set of A, and let $\mathfrak{R}$ denote the subring of $\ell^\infty(I)$ generated by the rows ρ_i of A and the point masses δ_i. If we let $A^t - I : \bigoplus_I \mathbb{Z} \to \mathfrak{R}$, then [8, Theorem 4.5] implies that $K_0(\mathcal{O}_A) \cong \mathrm{coker}(A^t - I)$ and $K_1(\mathcal{O}_A) \cong \ker(A^t - I)$. Now

$$A^t - I = \begin{pmatrix} 0&0&0&1&0&0&0 & \\ 0&0&0&0&1&0&0 & \\ 0&0&0&0&0&1&0 & \\ 1&1&1&0&0&0&1 & \cdots \\ 1&1&1&0&0&0&0 & \\ 1&1&1&0&0&0&0 & \\ 1&1&1&0&0&0&0 & \\ & & &\vdots & & & & \ddots \end{pmatrix}.$$

Let us examine $\ker(A^t - I)$. When $(A^t - I)(x_1, x_2, \ldots) = \vec{0}$, then

$$\begin{aligned} x_4 &= 0 \\ x_5 &= 0 \\ x_6 &= 0 \\ x_1 + x_2 + x_3 + x_7 &= 0 \\ x_1 + x_2 + x_3 + x_8 &= 0 \\ x_1 + x_2 + x_3 + x_9 &= 0 \\ &\vdots \end{aligned}$$

If $(x_1, x_2, \ldots) \in \bigoplus_I \mathbb{Z}$, then x_n is eventually zero, and the above equations reduce to $x_1 + x_2 + x_3 = 0$ and $x_i = 0$ for $i \geq 4$. Hence $\ker(A^t - I)$ is generated by $(-1, 1, 0, 0, 0, \ldots)$ and $(-1, 0, 1, 0, 0, \ldots)$ and $\ker(A^t - I)$ has rank 2. Thus $K_1(\mathcal{O}_A) \cong \ker(A^t - I) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Next we shall show that $A^t - I$ maps onto $\mathfrak{R}$. Since $\mathfrak{R}$ is a ring generated by $\{\rho_i, \delta_i\}$ we see that $\mathfrak{R}$ equals the collection of all sums of products of the ρ_i's and δ_i's. But for the matrix A above, any product of the ρ_i's and δ_i's may be written as a sum of $(0, 0, 0, 1, 1, 1, \ldots)$ and the δ_i's. Hence $\mathfrak{R} = \operatorname{span}_{\mathbb{Z}}\{(0, 0, 0, 1, 1, 1, \ldots), \delta_i : i \in I\}$. But, $(A^t - I)\delta_{i+3} = \delta_i$ and $(A^t - I)\delta_1 = (0, 0, 0, 1, 1, 1, \ldots)$, so $A^t - I$ maps onto $\mathfrak{R}$. Hence $K_0(\mathcal{O}_A) \cong \operatorname{coker}(A^t - I) \cong 0$. □

For the matrix A above, let $\mathcal{G} := (G^0, \mathcal{G}^1, r, s)$ be the ultragraph $\mathcal{G}_A$ of Definition 2.5. We define an ultragraph $\mathcal{F}$ by adding a single vertex $\{w\}$ to $\mathcal{G}$ and a countable number of edges with source w and range G^0. More precisely, we define $\mathcal{F} := (F^0, \mathcal{F}^1, r, s)$ by

$$F^0 := \{w\} \cup G^0 \qquad\qquad \mathcal{F}^1 := \{e_i\}_{i=1}^\infty \cup \mathcal{G}^1$$

and we extend r and s to $\mathcal{F}^1$ by defining $s(e_i) = \{w\}$ and $r(e_i) = G^0$ for all $1 \leq i < \infty$.

Note that $\mathcal{G}$ is unital because $G^0 \in \mathcal{G}^0$ [23, Lemma 3.2]. Since $r(e_i) = G^0 \in \mathcal{G}^A$ for all i we see from Lemma 2.2 that $\mathcal{F}^0 = \{A \cup \{w\} : A \in \mathcal{G}^0\} \cup \mathcal{G}^0$. It follows that $\mathcal{F}$ is also unital. Also note that $\mathcal{G}$ is transitive in the sense that $x \geq y$ for all $x, y \in G^0$.

Lemma 5.2. *Let $\mathcal{F}$ be the ultragraph described above and let $\mathcal{H} := \mathcal{G}^0$. Then $\mathcal{H}$ is a saturated hereditary subcollection of $\mathcal{F}^0$, the ideal $I_\mathcal{H} \triangleleft C^*(\mathcal{F})$ is Morita equivalent to $\mathcal{O}_A$, and $C^*(\mathcal{F})/I_\mathcal{H} \cong \mathbb{C}$.*

Proof. Let $\{s_e, p_A\}$ be the generating Cuntz-Krieger $\mathcal{F}$-family in $C^*(\mathcal{F})$. We shall first show that $I_\mathcal{H}$ is Morita equivalent to $C^*(\mathcal{G})$. Note that since $\mathcal{F}^0 = \{A \cup \{w\} : A \in \mathcal{G}^0\} \cup \mathcal{G}^0$, $\{s_e, p_A\}$ restricts to a Cuntz-Krieger $\mathcal{G}$-family. Now $I_\mathcal{H} = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{F}^* \text{ and } A \in \mathcal{G}^0\}$ by Lemma 3.5. If we let $p := p_{G^0}$, then $p \in I_\mathcal{H}$ and $p_{G^0} I_\mathcal{H} p_{G^0}$ is generated by $\{s_e, p_A : e \in \mathcal{G}^1 \text{ and } A \in \mathcal{G}^0\}$. Since $\mathcal{G}$ is a transitive ultragraph that is not a single loop, we see that $\mathcal{G}$ satisfies Condition (L). It then follows from the Cuntz-Krieger Uniqueness Theorem [23, Theorem 6.7] that $C^*(\mathcal{G}) \cong I_{\mathcal{H}}$. All that remains to show is that $pI_{\mathcal{H}}p$ is a full corner of $I_{\mathcal{H}}$. Suppose that J is an ideal in $I_{\mathcal{H}}$ containing $pI_{\mathcal{H}}p$. Since $p_{G^0}p_Ap_{G^0} = p_A$ for all $A \in \mathcal{G}^0$ we see that $\{p_A : A \in \mathcal{G}^0\} \subseteq J$. But then J contains the generators of $I_{\mathcal{H}}$ and $J = I$. Hence the corner is full and $I_{\mathcal{H}}$ is Morita equivalent to $C^*(\mathcal{G})$.

We shall now show that $C^*(\mathcal{F})/I_{\mathcal{H}} \cong \mathbb{C}$. To do this we shall first show that $p_w \notin I_{\mathcal{H}}$. If it was the case that $p_w \in I_{\mathcal{H}} = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{F}^1, A \in \mathcal{F}^0\}$, then we could find a linear combination such that

$$\|p_w - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\| < 1.$$

Also since

$$\|p_w\Big(p_w - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\Big)\| \leq \|p_w - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\|$$

we may assume that $|\alpha| \geq 1$ and $s(\alpha_k) \in r(e)$. Let F be the (necessarily finite) set of edges that are the initial edge of an α_i. Because w is an infinite emitter, it follows that $q := p_w - \sum_{e \in F} s_e s_e^*$ is a nonzero projection. Hence

$$\|p_w - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\| \geq \|q\Big(p_w - \sum_{k=1}^n \lambda_k s_{\alpha_k} p_{A_k} s_{\beta_k}^*\Big)\| = \|q\| = 1$$

which is a contradiction. Therefore $p_w \notin \mathcal{H}$, and $C^*(\mathcal{F})/I_{\mathcal{H}}$ is generated by the projection $p_w + I_{\mathcal{H}}$. Consequently, $C^*(\mathcal{F})/I_{\mathcal{H}} \cong \mathbb{C}$. □

The ideas in the proof of the following proposition were suggested by Wojciech Szymański.

Proposition 5.3. *The ultragraph algebra $C^*(\mathcal{F})$ is not an Exel-Laca algebra.*

Proof. Recall that a character for $C^*(\mathcal{F})$ is a nonzero homomorphism $\epsilon : C^*(\mathcal{F}) \to \mathbb{C}$. We shall show that there is a unique character on $C^*(\mathcal{F})$. Let $\{s_e, p_A\}$ be a generating Cuntz-Krieger $\mathcal{F}$-family.

Since $C^*(\mathcal{F})/I_{\mathcal{H}} \cong \mathbb{C}$ by Lemma 5.2 we see that the projection $\pi : C^*(\mathcal{F}) \to C^*(\mathcal{F})/I_{\mathcal{H}}$ is a character. We shall now show that this character is unique. Let $\epsilon : C^*(\mathcal{F}) \to \mathbb{C}$ be a character. Set $I = \ker \epsilon$. Then I is a nonzero ideal and $\mathcal{H} := \{A \in \mathcal{F}^0 : p_A \in I\}$ is a saturated hereditary subcollection. Since $\mathcal{G}$ is transitive, we see that $\mathcal{F}$ satisfies Condition (L). Therefore, the Cuntz-Krieger Uniqueness Theorem [23, Theorem 6.7] implies that $\ker \epsilon$ contains one of the p_A's, and $\mathcal{H}$ is nonempty. Because $\mathcal{H}$ is nonempty and $\mathcal{G}$ is transitive, it follows that $\mathcal{G}^0 \subseteq \mathcal{H}$. Now since ϵ is nonzero, we cannot also have $\{w\}$ in $\mathcal{H}$. Therefore, $\mathcal{H} = \mathcal{G}^0$, and this implies that $p_v \in I$ for all $v \in G^0$ and $s_e = s_e p_{r(e)} \in I$ for all $e \in \mathcal{F}^1$. Since $C^*(\mathcal{F})$ is generated by $\{s_e : e \in \mathcal{F}^1\} \cup \{p_v : v \in F^0 = G^0 \cup \{w\}\}$, and

$$\epsilon(p_w) = \epsilon(s_e) = 0 \qquad \text{for all } w \in G^0 \text{ and } e \in \mathcal{F}^1$$

we see that ϵ is completely determined by its value on p_w. Because p_w is a projection, $\epsilon(p_w) = 1$. Thus ϵ is unique.

Now if $C^*(\mathcal{F})$ was an Exel-Laca algebra, then $C^*(\mathcal{F})$ would be generated by an Exel-Laca family $\{S_i\}$. Let γ be the gauge action on this Exel-Laca algebra. Because there is a unique character ϵ on $C^*(\mathcal{F})$, we see that $\epsilon \circ \gamma_z = \epsilon$ for all $z \in \mathbb{T}$.

Also, since ϵ is nonzero, $\epsilon(S_i) \neq 0$ for some i. Thus $\epsilon(S_i) = \epsilon(\gamma_z(S_i)) = z\epsilon(S_i)$ for all $z \in \mathbb{T}$ which is a contradiction. □

Proposition 5.4. *The ultragraph algebra $C^*(\mathcal{F})$ is not a graph algebra.*

Proof. Let $\mathcal{H} := \mathcal{G}^0$. Then $\mathcal{H}$ is a saturated hereditary subcollection of $\mathcal{F}^0$. The short exact sequence $0 \to I_{\mathcal{H}} \to C^*(\mathcal{F}) \to C^*(\mathcal{F})/I_{\mathcal{H}} \to 0$ induces the following cyclic six term exact sequence for K-theory:

$$\begin{array}{ccccc} K_0(I_{\mathcal{H}}) & \longrightarrow & K_0(C^*(\mathcal{F})) & \longrightarrow & K_0(C^*(\mathcal{F})/I_{\mathcal{H}}) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathcal{F})/I_{\mathcal{H}}) & \longleftarrow & K_1(C^*(\mathcal{F})) & \longleftarrow & K_1(I_{\mathcal{H}}) \end{array}$$

It follows from Lemma 5.2 that $C^*(\mathcal{F})/I_{\mathcal{H}} \cong \mathbb{C}$. Thus $K_0(C^*(\mathcal{F})/I_{\mathcal{H}}) \cong \mathbb{Z}$ and $K_1(C^*(\mathcal{F})/I_{\mathcal{H}}) \cong 0$. Also, Lemma 5.2 tells us that $I_{\mathcal{H}}$ is Morita equivalent to $\mathcal{O}_A$, and it then follows from Lemma 5.1 that $K_0(I_{\mathcal{H}}) \cong 0$ and $K_1(I_{\mathcal{H}}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus the above exact sequence becomes

$$0 \longrightarrow K_0(C^*(\mathcal{F})) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow K_1(C^*(\mathcal{F})) \longrightarrow 0$$

and $\operatorname{rank} K_0(C^*(\mathcal{F})) < \operatorname{rank} K_1(C^*(\mathcal{F}))$.

Now we see that $F^0 = G^0 \cup \{w\} \in \mathcal{F}^0$ and thus $C^*(\mathcal{F})$ is unital. Therefore, if $C^*(\mathcal{F})$ were the C^*-algebra of a graph, then this graph would have to have a finite number of vertices. It then follows from [19, Theorem 3.2] that there exists an exact sequence

$$0 \longrightarrow K_1(C^*(\mathcal{F})) \longrightarrow \bigoplus_V \mathbb{Z} \longrightarrow \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \longrightarrow K_0(C^*(\mathcal{F})) \longrightarrow 0$$

for some finite sets V and W. Hence $\operatorname{rank} K_1(C^*(\mathcal{F})) \leq \operatorname{rank} K_0(C^*(\mathcal{F}))$, which is a contradiction. □

Corollary 5.5. *If $\mathcal{F}$ is the ultragraph described above, then $C^*(\mathcal{F})$ is neither an Exel-Laca algebra nor a graph algebra.*

6. Viewing Ultragraph Algebras as Cuntz-Pimsner Algebras

Let X be a Hilbert bimodule over a C^*-algebra $\mathcal{A}$, in the sense that X is a right Hilbert $\mathcal{A}$-module with a left action of $\mathcal{A}$ by adjointable operators. In [18] Pimsner described how to construct a C^*-algebra $\mathcal{O}_X$ from X. These Cuntz-Pimsner algebras have been shown to include many classes of C^*-algebras and consequently have been the subject of much attention. Pimsner originally showed that for appropriate choices of X and $\mathcal{A}$, the Cuntz-Pimsner algebras included the Cuntz-Krieger algebras [18, §1 Example 2] as well as crossed products by $\mathbb{Z}$ [18, §1 Example 3]. Since that time it has also been shown that the C^*-algebras of graphs with no sinks [9, Proposition 12] and the Exel-Laca algebras [21, Theorem 5] may be realized as Cuntz-Pimsner algebras.

In this section we show that the C^*-algebras of ultragraphs with no sinks may also be realized as Cuntz-Pimsner algebras using a construction similar to that in [21]. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph with no sinks. Define $\mathcal{A}$ to be the C^*-subalgebra of $C^*(\mathcal{G})$ generated by $\{p_A : A \in \mathcal{G}^0\}$. Note that since the p_A's commute and $\{p_A : A \in \mathcal{G}^0\}$ is closed under multiplication, it follows that $\mathcal{A} = \overline{\operatorname{span}}\{p_A : A \in \mathcal{G}^0\}$. Also let $X := \overline{\operatorname{span}}\{s_e p_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$. Then X

has a natural Hilbert $\mathcal{A}$-bimodule structure with the right action given by right multiplication, the left action given by left multiplication, and the $\mathcal{A}$-valued inner product given by $\langle x, y\rangle_{\mathcal{A}} := x^*y$.

We shall let $\phi : \mathcal{A} \to \mathcal{L}(X)$ denote the map given by the left action; that is, $\phi(a)(x) := ax$. We shall also let $\mathcal{K}(X)$ denote the compact operators on X and $J(X) := \phi^{-1}(\mathcal{K}(X))$.

Theorem 6.1. *If X is the Hilbert bimodule defined above, then $\mathcal{O}_X$ is canonically isomorphic to $C^*(\mathcal{G})$.*

Proof. Using the language of [10], let $(k_X, k_{\mathcal{A}})$ be a universal Toeplitz representation of X in $\mathcal{O}_X$ which is Cuntz-Pimsner covariant (i.e. coisometric on $J(X)$). We shall show that $\{k_X(s_e), k_{\mathcal{A}}(p_A)\}$ is a Cuntz-Krieger $\mathcal{G}$-family in $\mathcal{O}_X$.

Since $k_{\mathcal{A}}$ is a homomorphism, we trivially have $k_{\mathcal{A}}(p_Ap_B) = k_{\mathcal{A}}(p_A)k_{\mathcal{A}}(p_B)$ and $k_{\mathcal{A}}(p_{A\cup B}) = k_{\mathcal{A}}(p_A) + k_{\mathcal{A}}(p_B) - k_{\mathcal{A}}(p_{A\cap B})$. Because $(k_X, k_{\mathcal{A}})$ is a Toeplitz representation we have $k_X(s_e)^*k_X(s_e) = k_{\mathcal{A}}(\langle s_e, s_e\rangle_{\mathcal{A}}) = k_{\mathcal{A}}(s_e^*s_e) = k_{\mathcal{A}}(p_{r(e)})$. Also $k_{\mathcal{A}}(p_{s(e)})k_X(s_e) = k_X(p_{s(e)}s_e) = k_X(s_e)$ so $k_X(s_e)k_X(s_e)^* \leq k_{\mathcal{A}}(p_{s(e)})$. Finally, if v is the source of finitely many vertices, then $p_v = \sum_{s(e)=v} s_es_e^*$ and $\phi(p_v) = \sum_{s(e)=v} \Theta_{s_e,s_e}$. It then follows from the fact that $(k_X, k_{\mathcal{A}})$ is Cuntz-Pimsner covariant that $k_{\mathcal{A}}(p_v) = k_{\mathcal{A}}^{(1)}(\phi(p_v)) = k_{\mathcal{A}}^{(1)}(\sum_{s(e)=v} \Theta_{s_e,s_e}) = \sum_{s(e)=v} k_X(s_e)k_X(s_e)^*$. Hence $\{k_X(s_e), k_{\mathcal{A}}(p_A)\}$ is a Cuntz-Krieger $\mathcal{G}$-family and the universal property of $C^*(\mathcal{G})$ gives a homomorphism $\Phi : C^*(\mathcal{G}) \to \mathcal{O}_X$ with $\Phi(s_e) = k_X(s_e)$ and $\Phi(p_A) = k_{\mathcal{A}}(p_A)$.

Let $\psi : X \hookrightarrow C^*(\mathcal{G})$ and $\pi : \mathcal{A} \hookrightarrow C^*(\mathcal{G})$ be the inclusion maps. Then (ψ, π) is a Toeplitz representation. To see that (ψ, π) is also Cuntz-Pimsner covariant, let $a \in \mathcal{A}$ with $\phi(a) \in \mathcal{K}(X)$. Then $\phi(a) = \lim \sum \lambda_k\Theta_{x_k,y_k}$ and hence $a = \lim \sum \lambda_k x_k^* y_k$. But then

$$\begin{aligned}\pi^{(1)}(\phi(a)) &= \lim \sum \lambda_k \pi^{(1)}(\Theta_{x_k,y_k}) = \lim \sum \lambda_k \psi(x_k)^*\psi(y_k) \\ &= \lim \sum \lambda_k x_k^* y_k = a = \pi(a).\end{aligned}$$

Since (ψ, π) is a Toeplitz representation which is Cuntz-Pimsner covariant, the universal property of $\mathcal{O}_X$ [10, Proposition 1.3] implies that there is a homomorphism $\Phi' : \mathcal{O}_X \to C^*(G)$ which commutes with (ψ, π). But then $\Phi'(k_{\mathcal{A}}(p_A)) = \pi(p_A) = p_A$ and $\Phi'(k_X(s_e)) = \psi(s_e) = s_e$ so Φ and Φ' are inverses for each other. □

In the remainder of this section we shall give a description of $J(X) := \phi^{-1}(\mathcal{K}(X))$ in terms of the ultragraph.

Lemma 6.2. *If X is the Hilbert bimodule of an ultragraph $\mathcal{G}$, then $\phi(a) \in \mathcal{K}(X)$ implies $a \in \overline{\operatorname{span}}\{s_ep_As_f^* : e, f \in \mathcal{G}^1, A \in \mathcal{G}^0\}$.*

Proof. Since $X := \overline{\operatorname{span}}\{s_ep_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ we have $\mathcal{K}(X) = \overline{\operatorname{span}}\{\Theta_{s_ep_A,s_fp_B} : e, f \in \mathcal{G}^1 A, B \in \mathcal{G}^0\}$. If $\phi(a) \in \mathcal{K}(X)$, then for any $\epsilon > 0$ there exists a finite linear combination with $\|\phi(a) - \sum \Theta_{s_ep_A,s_fp_B}\| < \epsilon$. Thus

$$\|\phi(a)(x) - \sum \Theta_{s_ep_A,s_fp_B}(x)\| = \|ax - \sum s_ep_{A\cap B}s_f^*x\| < \epsilon$$

for all $x \in X$. Hence $\|a - \sum s_ep_{A\cap B}s_f^*\| < \epsilon$ and the claim is proven. □

Throughout the following whenever $B, C \in \mathcal{G}^0$ we shall let $Q(B, C) := p_B - p_Bp_C$.

Lemma 6.3. *If $A_1, \dots, A_n \in \mathcal{G}^0$, then*
$$\sum_{I \subseteq \{1,\dots n\}} Q(\cap_{i\in I} A_i, \cup_{i \notin I} A_i) = 1.$$

Proof. Induct on n. Multiply the formula for $n=k$ by $p_{A_{k+1}} + (1 - p_{A_{k+1}})$. □

Lemma 6.4. *If $\sum_{k=1}^n \lambda_k p_{A_k}$ is a finite linear combination with $A_k \in \mathcal{G}^0$ for all k, then*
$$\sum_{k=1}^n \lambda_k p_{A_k} = \sum_{I \subseteq \{1,\dots,n\}} a_I Q(\cap_{i\in I} A_i, \cup_{i \notin I} A_i)$$
where $a_I := \sum_{i \in I} \lambda_i$. Consequently $\sum_{k=1}^n \lambda_k p_{A_k}$ can be rewritten as a linear combination of mutually orthogonal projections of the form $Q(B,C)$ with $B, C \in \mathcal{G}^0$.

Proof. For convenience of notation, let $N_n := \{1, \dots n\}$. We shall prove the claim by induction on n. For $n = 1$ the equality holds easily. Therefore, assume the equality is true for n and we shall prove it for $n+1$.

$$\begin{aligned}
&\sum_{I \subseteq N_{n+1}} a_I Q(\bigcap_{i \in I} A_i, \bigcup_{i \in N_{n+1} \setminus I} A_i) \\
&= \sum_{I \subseteq N_n} a_I Q(\bigcap_{i \in I} A_i, \bigcup_{i \in N_{n+1} \setminus I} A_i) + \sum_{I \subseteq N_n} (a_I + \lambda_{n+1}) Q(A_{n+1} \cap \bigcap_{i \in I} A_i, \bigcup_{i \in N_n \setminus I} A_i) \\
&= \sum_{I \subseteq N_n} a_I p_{\cap_{i\in I} A_i} - a_I p_{\cap_{i \in I} A_i} p_{\cup_{i \in N_{n+1} \setminus I} A_i} \\
&\qquad + \sum_{I \subseteq N_n} (a_I + \lambda_{n+1}) (p_{\cap_{i\in I} A_i} p_{A_{n+1}} - p_{\cap_{i \in I} A_i} p_{A_{n+1}} p_{\cup_{i \in N_n \setminus I} A_i}) \\
&= \sum_{I \subseteq N_n} a_I p_{\cap_{i\in I} A_i} - a_I p_{\cap_{i \in I} A_i} (p_{\cup_{i \in N_n \setminus I} A_i} + p_{A_{n+1}} - p_{A_{n+1}} p_{\cup_{i \in N_n \setminus I} A_i}) \\
&\qquad + \sum_{I \subseteq N_n} (a_I + \lambda_{n+1}) (p_{\cap_{i\in I} A_i} p_{A_{n+1}} - p_{\cap_{i \in I} A_i} p_{A_{n+1}} p_{\cup_{i \in N_n \setminus I} A_i}) \\
&= \sum_{I \subseteq N_n} a_I p_{\cap_{i\in I} A_i} - a_I p_{\cap_{i \in I} A_i} p_{\cup_{i \in N_n \setminus I} A_i} \\
&\qquad + \sum_{I \subseteq N_n} \lambda_{n+1} (p_{\cap_{i\in I} A_i} p_{A_{n+1}} - p_{\cap_{i \in I} A_i} p_{A_{n+1}} p_{\cup_{i \in N_n \setminus I} A_i}) \\
&= \sum_{k=1}^n \lambda_k p_{A_k} + \lambda_{n+1} p_{A_{n+1}} \sum_{I \subseteq N_n} (p_{\cap_{i \in I} A_i} - p_{\cap_{i\in I} A_i} p_{\cup_{i \in N_n \setminus I} A_i}) \\
&= \sum_{k=1}^{n+1} \lambda_k p_{A_k}
\end{aligned}$$

where this last line follows from Lemma 6.3.

The final claim follows from the fact that the terms $Q(\cap_{i\in I} A_i, \cup_{i\notin I} A_i)$ and $Q(\cap_{i \in J} A_i, \cup_{i \notin J} A_i)$ are orthogonal when $I \neq J$. □

Proposition 6.5. *If $\mathcal{G}$ is an ultragraph with no sinks and X is the Hilbert bimodule defined above, then*
$$\phi^{-1}(\mathcal{K}(X)) = \overline{\operatorname{span}}\{p_v : v \in G^0 \text{ and } v \text{ is not an infinite emitter}\}.$$

Proof. Let I denote the right hand side of the above equation. If $v \in G^0$ is not an infinite emitter, then $p_v = \sum_{s(e)=v} s_e s_e^*$ and $\phi(p_v) = \sum_{s(e)=v} \Theta_{s_e,s_e} \in \mathcal{K}(X)$. Hence $I \subseteq \phi^{-1}(\mathcal{K}(X))$.

To see the reverse inclusion let $a \in \mathcal{A}$ and $\phi(a) \in \mathcal{K}(X)$. Choose $\epsilon > 0$. Since $\mathcal{A} = \overline{\operatorname{span}}\{p_A : A \in \mathcal{G}^0\}$, Lemma 6.4 implies that there exists a finite linear combination $\sum_{k=1}^n \lambda_k Q(B_k, C_k)$ with the $Q(B_k, C_k)$'s mutually orthogonal and $\|a - \sum_{k=1}^n \lambda_k Q(B_k, C_k)\| < \epsilon/2$. Define $s^{-1}(B \backslash C) := \{e \in \mathcal{G}^1 : s(e) \in B \backslash C\}$, and let $S_1 := \{k : |s^{-1}(B_k \backslash C_k)| < \infty\}$ and $S_2 := \{k : |s^{-1}(B_k \backslash C_k)| = \infty\}$. Let $\lambda_{k'} := \max\{|\lambda_k| : k \in S_2\}$. Because $\phi(a) \in \mathcal{K}(X)$, we know from Lemma 6.2 that $a \in \overline{\operatorname{span}}\{s_e p_A s_f^* : e, f \in \mathcal{G}^1, A \in \mathcal{G}^0\}$. Thus for every $\epsilon' > 0$ we may find a finite linear combination $\sum_{j=1}^m \mu_j s_{e_j} p_{A_j} s_{f_j}^*$ such that $\|a - \sum_{j=1}^m \mu_j s_{e_j} p_{A_j} s_{f_j}^*\| < \epsilon'$.

Now since $|s^{-1}(B_{k'} \backslash C_{k'})| = \infty$, there exists an edge g such that $s(g) \in B_{k'} \backslash C_{k'}$ and g is not equal to any of the f_j's. Since the $Q(B_k, C_k)$'s are mutually orthogonal, we have that $s(g) \notin B_k \backslash C_k$ for all $k \neq k'$. Hence

$$\begin{aligned}
|\lambda_{k'}| &= \|\lambda_{k'} Q(B_{k'}, C_{k'})\| \\
&= \|\sum_{k=1}^n \lambda_k Q(B_k, C_k) s_g - \sum_{j=1}^m \mu_j s_{e_j} p_{A_j} s_{f_j}^* s_g\| \\
&\leq \|\sum_{k=1}^n \lambda_k Q(B_k, C_k) - \sum_{j=1}^m \mu_j s_{e_j} p_{A_j} s_{f_j}^*\| \; \|s_g\| \\
&\leq \|\sum_{k=1}^n \lambda_k Q(B_k, C_k) - a\| + \|a - \sum_{j=1}^m \mu_j s_{e_j} p_{A_j} s_{f_j}^*\| \\
&\leq \|\sum_{k=1}^n \lambda_k Q(B_k, C_k) - a\| + \epsilon'.
\end{aligned}$$

Since this inequality holds for all $\epsilon' > 0$ we have $|\lambda_{k'}| \leq \|\sum_{k=1}^n \lambda_k Q(B_k, C_k) - a\| < \epsilon/2$. Thus

$$\begin{aligned}
\|a - \sum_{k \in S_1} \lambda_k Q(B_k, C_k)\| &\leq \|a - \sum_{k=1}^n \lambda_k Q(B_k, C_k)\| + \|\sum_{k \in S_2} \lambda_k Q(B_k, C_k)\| \\
&< \frac{\epsilon}{2} + |\lambda_{k'}| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now for every $k \in S_1$ we have that $s^{-1}(B_k \backslash C_k) < \infty$. Since $\mathcal{G}$ has no sinks, this implies that $B_k \backslash C_k$ is the union of a finite number of vertices that emit finitely many edges. Since $B_k \backslash C_k$ is finite [23, Lemma 4.2] implies that $p_{B_k \backslash C_k} = \sum_{v \in B_k \backslash C_k} p_v$. Furthermore, since $B_k \backslash C_k$ is finite, it is an element of $\mathcal{G}^0$ and the equality $p_{B_k} = p_{B_k \backslash C_k} + p_{B_k \cap C_k} - p_\emptyset$ shows that $Q(B_k, C_k) = p_{B_k \backslash C_k} = \sum_{v \in B_k \backslash C_k} p_v$. Since the p_v's all emit finitely many edges and since ϵ was arbitrary, the above shows that $a \in \overline{\operatorname{span}}\{p_v : v \in G^0$ and v is not an infinite emitter$\}$ and hence $\phi^{-1}(\mathcal{K}(X)) \subseteq I$. □

Corollary 6.6. *If $\mathcal{G}$ is an ultragraph with no sinks, then $\phi^{-1}(\mathcal{K}(X)) \cong C_0(T)$ where $T := \{v \in G^0 : v$ emits finitely many edges$\}$ has the discrete topology.*

Proof. Let $\delta_v \in C_0(T)$ denote the point mass at v. Then the map $\delta_v \mapsto p_v$ extends to an isomorphism from $C_0(T)$ onto $\overline{\operatorname{span}}\{p_v : v$ emits finitely many edges$\}$. □

Remark 6.7. If G is a graph, then the C^*-algebra $\mathcal{A}$ in the graph bimodule [11, Example 1.2] is equal to $C_0(G^0)$ for the discrete space G^0. For an ultragraph $\mathcal{G}$ the C^*-algebra $\mathcal{A}$ arising in the ultragraph bimodule is the C^*-algebra generated by $\{p_A : A \in \mathcal{G}^0\}$. Since the p_A's commute $\mathcal{A}$ is commutative and $\mathcal{A} \cong C_0(X)$ for some locally compact space X. Furthermore, since $\mathcal{A}$ is generated by projections the space X must be totally disconnected. However, in general X need not be discrete. Despite this, the above corollary shows that the ideal $\phi^{-1}(\mathcal{K}(X))$ corresponds to $C_0(T)$ for some discrete open set $T \subseteq X$.

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