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Moore Cohomology, Principal Bundles, and Actions of Groups on c*-algebras

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Introduction. In recent years both topological and algebraic invariants have been associated to group actions on C^*-algebras. Principal bundles have been used to describe the topological structure of the spectrum of crossed products [18, 19], and as a result we now know that crossed products of even the very nicest non-commutative algebras can be substantially more complicated than those of commutative algebras [19, 5]. The algebraic approach involves group cohomological invariants, and exploits the associated machinery to classify group actions on C^*-algebras; this originated in [2], and has been particularly successful for actions of $\mathbf{R}$ and tori ([19; Section 4], [21]). Here we shall look in detail at the relationship between these topological and algebraic invariants, with a view to analyzing the structure of the systems studied in [19; Section 2, 3].

Our starting point is a theorem of Rosenberg [21, Theorem 2.5] concerning the locally unitary actions of [18]. If A has Hausdorff spectrum X, and $\alpha : G \to \mathrm{Aut}(A)$ is an action of an abelian group which is locally implemented by homomorphisms $u : G \to \mathcal{UM}(A)$, then the spectrum of the crossed product $A \rtimes_\alpha G$ is a principal $\hat{G}$-bundle over X; the class $\zeta(\alpha)$ of the bundle determines α up to exterior equivalence, and all such bundles arise [18]. If G is connected, the range of α will often lie in the subgroup $\mathrm{Inn}(A)$ of inner automorphisms, and then there is a class $c(\alpha)$ in the Moore cohomology group $H^2(G,C(X,\mathbf{T}))$ which is trivial when evaluated at points of X, and which vanishes precisely when α is implemented by a unitary group u [19, Section 0]. Rosenberg showed how to construct a principal bundle directly from a pointwise trivial class in $H^2(G,C(X,\mathbf{T}))$, and that his construction connects up the invariants $c(\alpha)$ and $\zeta(\alpha)$.

We aim to prove a version of Rosenberg's theorem for actions $\alpha : G \to \mathrm{Aut}(A)$ which are locally unitary on a subgroup N. It was shown in [19; Section 2] that, provided $X = \hat{A}$ is a principal G/N-bundle, there is a commutative

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diamond of principal bundles describing the spectrum of $A \rtimes_\alpha G$ as a principal $\hat{N}$-bundle over X/G:

$$\begin{tikzcd}
& ((A \rtimes_\alpha N))^\wedge \arrow[dl, "\mathrm{Ind}"'] \arrow[dr, "\mathrm{Res}"] & \\
((A \rtimes_\alpha G))^\wedge \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\
& X/G &
\end{tikzcd}$$

When $\alpha(N) \subseteq \mathrm{Inn}(A)$, we shall associate to α an invariant $d(\alpha)$ lying in a relative Moore cohomology group $\Lambda\big(G,N;C(X,\mathbf{T})\big)$, which vanishes exactly when there is a homomorphism $u : N \to \mathcal{UM}(A)$ such that (A,G,α,u) is a twisted system in the sense of Green. We shall then show how to construct diamonds of principal bundles directly from appropriate elements of $\Lambda\big(G,N;C(X,\mathbf{T})\big)$, in such a way that the element vanishes if and only if the bottom left-hand arrow is a trivial bundle. Since applying our construction to $d(\alpha)$ gives the diamond describing $(A \rtimes_\alpha G)^\wedge$, our main result follows immediately:

Theorem. *Let (A,G,α) be a separable C^*-dynamical system in which G is abelian and the spectrum X of A is Hausdorff. Suppose N is a closed subgroup of G such that $\alpha|_N$ is locally unitary and $X \to X/G$ is a locally trivial principal G/N-bundle. Then $(A \rtimes_\alpha G)^\wedge$ is trivial as an $\hat{N}$-bundle over X/G if and only if α is given on N by a Green twisting map $u : N \to \mathcal{UM}(A)$ (i.e., u satisfies $\alpha|_N = \mathrm{Ad}\, u$ and $\alpha_s(u_n) = u_n$ for $s \in G$, $n \in N$).*

This result has some interesting corollaries. First of all, if (A,G,α,u) is a twisted system then a theorem of Olesen and Pedersen [12] says that $A \rtimes_\alpha G$ is isomorphic to an induced algebra $\mathrm{Ind}_{N^\perp}^{\hat{G}}(B,\vartheta)$—indeed, we can take B to be the restricted crossed product $A \rtimes_{\alpha,N}^{u} G$, and ϑ to be the dual action of $N^\perp = (G/N)^\wedge$. Thus, in our setting, the triviality of $(A \rtimes_\alpha G)^\wedge$ as an $\hat{N}$-bundle implies that $A \rtimes_\alpha G$ is an induced C^*-algebra. If we have a system (A,G,α) as in the theorem and $X \to X/G$ is trivial as a G/N-bundle, then we can apply this reasoning to the dual system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$, and deduce that α is the translation action on an induced algebra. Even if the bundle $X \to X/G$ is only locally trivial, we can still apply this reasoning locally on G-invariant subsets of X, and the resulting local structure theorem for the systems studied in [19, Section 2] should have some very interesting consequences. In particular, we believe

that using it we can identify exactly which commutative diamonds of principal bundles arise from these actions, and we intend to pursue this in the near future.

Although we have so far discussed locally unitary actions and locally trivial principal bundles, it is perhaps more natural to consider pointwise unitary actions, which are easily described as actions which fix the spectrum and whose Mackey obstructions vanish. It was shown in [13] that the results of [18] and [19] remain true for pointwise unitary actions, provided we restrict attention to continuous-trace algebras and use free and proper actions instead of locally trivial bundles. We shall therefore be studying systems (A, G, α) in which A has continuous trace, $\alpha|_N$ is pointwise unitary and G/N acts freely and properly on $\hat{A}$. We do need to assume A has continuous trace to ensure $(A \rtimes_\alpha G)^\wedge$ is Hausdorff ([13, Theorem 1.7]; see also [13, Section 2(c)]); however, this is also true if α is locally unitary and $\hat{A}$ is Hausdorff, and our arguments will apply also in this case, giving the theorem stated above.

We begin with a short section about free and proper G-spaces, in which we set up our terminology—we call them G-bundles, and reserve the word "principal" for locally trivial ones—and prove a couple of routine lemmas. In Section 3 we discuss Rosenberg's construction of a $\hat{G}$-bundle E_w from a pointwise trivial element w of $H^2(G, C(X, \mathbf{T}))$, presenting it so that it works also for arbitrary locally compact abelian groups. We prove that $w \mapsto E_w$ is an injection of the pointwise trivial part of $H^2(G, C(X, \mathbf{T}))$ into the group of $\hat{G}$-bundles over X. Rosenberg's theorem asserts that this map is surjective when G is connected, but this is not the case when G is not connected, and we begin the identification of the range in Corollary 3.10. The methods of Section 3 are direct, making no mention of operator algebras, but in Section 4 we connect them up with group actions on continuous-trace algebras. In particular, using C^*-algebraic methods it is easy to identify the range of the map $w \mapsto E_w$, although this is also done directly in Section 8.

We introduce our relative cohomology group Λ and the invariant $d(\alpha)$ in Section 5. The group Λ is a Borel version of one which is used by algebraists to classify crossed extensions (e.g., [4, 20]), and which has also appeared in classifications of discrete group actions on injective von Neumann algebras (e.g., [6, 25]); we discuss this further in Remark 5.4. The invariant $d(\alpha)$ lies in a subgroup Λ_{PT} of appropriately pointwise trivial elements, and in Section 6 we show how to associate to each element (λ, μ) of $\Lambda_{PT}(G, N; C(X, \mathbf{T}))$ a commutative diamond of bundles, and in particular an $\hat{N}$-bundle $q : F_{(\lambda,\mu)} \to X/G$. Our extension of Rosenberg's theorem asserts that $(\lambda, \mu) \mapsto F_{(\lambda,\mu)}$ is an injection of Λ_{PT} into the group of $\hat{N}$-bundles over X/G, and identifies the range; we defer the proof of the last part to Section 8 since we do not need it for our applications to C^*-algebras. The results in Section 6 lean heavily on the version of Rosenberg's construction given in Section 3.

We prove our main theorem and its corollaries in Section 7. As we mentioned above, it follows almost immediately from our earlier results, and indeed this is the point we wish to make. Here we have two natural invariants, the topological one arising concretely as the spectrum of the crossed product, the algebraic one directly measuring the obstruction to solving our problem, and it is the relationship between them which is giving the information we want. Thus while a direct proof would also be possible, we have preferred to stress the connection between the two invariants.

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2. Free and proper G-spaces. As we mentioned in the introduction, we shall refer to free and proper G-spaces as G-bundles, and locally trivial principal G-bundles as principal G-bundles. In this section we shall make this precise, show how to multiply G-bundles if G is abelian, and see what happens when we change the structure group G.

Definition 2.1. Let G be a locally compact group and T a locally compact (Hausdorff) space. A *G-bundle* over T is a locally compact space X, together with a free and proper action of G on X and a continuous G-invariant map $p : X \to T$ which induces a homeomorphism of X/G onto T. Two such G-bundles $p : X \to T$ and $q : Y \to T$ are *isomorphic* if there is a G-equivariant homeomorphism h of X onto Y such that $q \circ h = p$; we denote by $HP(T, \mathcal{G})$ the set of isomorphism classes of G-bundles over T. (This notation is motivated by the connection with sheaf cohomology; see Remark 2.5 below.)

Remark 2.2. A G-bundle $p : X \to T$ is *trivial* if it is isomorphic to $T \times G$, or, equivalently, if there is a continuous section $s : T \to X$ for p. Similarly, a G-bundle is *locally trivial* if p has local sections; we shall call a locally trivial G-bundle a *principal G-bundle.* It follows from Palais's slice theorem [16; Theorem 4.1] that if G is a Lie group, then every G-bundle is a principal G-bundle.

Remark 2.3. If $p : X \to T$ and $q : Y \to T$ are G-bundles, and if $h : X \to Y$ is a G-equivariant map satisfying $q \circ h = p$, then h must be bijective and Lemma 1.12 of [13] implies that h is an isomorphism of G-bundles.

Lemma 2.4. *Suppose that G is a locally compact abelian group, and that $p : X \to T$ and $q : Y \to T$ are G-bundles. Then the action of G on the fibre product*

$$X \overset{\times}{_T} Y = \{(x,y) \in X \times Y : p(x) = q(x)\}$$

defined by $s \cdot (x,y) = (s \cdot x, s^{-1} \cdot y)$ is free and proper. The formula $s \cdot [x,y] = [s \cdot x, y]$ defines a free and proper action of G on the quotient space

$$X * Y = (X \overset{\times}{_T} Y)/G,$$

and, together with the map $r : X * Y \to T$ sending $[x,y]$ to $p(x)$, makes $X * Y$ into a G-bundle over T. The binary operation $[X][Y] = [X * Y]$ is well-defined on $HP(T,\mathcal{G})$, and with it $HP(T,\mathcal{G})$ becomes an abelian group.

Proof. Most of the details are routine, so we shall omit them. We should observe, however, that the identity in $HP(T,\mathcal{G})$ is the class $[T \times G]$ of the trivial bundle, and that the inverse of $[X]$ is the class $[X^{\mathrm{op}}]$ of the opposite G-space X^{op}. (Recall that X^{op} equals X as a set, but carries the G-action given by $s * x = s^{-1} \cdot x$.) To see that $X * X^{\mathrm{op}}$ is trivial, notice that the map $x \mapsto [x,x]$ is a continuous G-invariant map from X to $X * X^{\mathrm{op}}$, and hence defines a continuous section from X/G to $X * X^{\mathrm{op}}$; the claim follows from Remark 2.2. $\square$

Remark 2.5. As usual, if we denote by $\mathcal{G}$ the sheaf of germs of continuous G valued functions, we can use transition functions to identify the group of isomorphism classes of locally trivial G-bundles with the sheaf cohomology group $\check{H}^1(T,\mathcal{G})$, and hence realize $\check{H}^1(T,\mathcal{G})$ as a subgroup of $HP(T,\mathcal{G})$ (see Remark 2.7). It is well-known that $\check{H}^1(T,\mathcal{G})$ can be a proper subgroup of $HP(T,\mathcal{G})$; for example, let $G = \prod_{n=1}^{\infty} \mathbf{Z}_2$, viewed as a subgroup of $X = T = \prod_{n=1}^{\infty} \mathbf{T}$, and define $p : X \to T$ by $p(\{z_n\}) = \{z_n^2\}$. Then $G = \ker p$ acts freely and properly on X, but p cannot have local sections since any open set in T contains one of the form $\left(\prod_{n=1}^{N} W_n\right) \times \left(\prod_{n=N+1}^{\infty} \mathbf{T}\right)$, and a section defined on this set would, in particular, give a section for the map $z_{N+1} \mapsto z_{N+1}^2$.

Proposition 2.6. *Suppose that $\varphi : G \to H$ is a continuous homomorphism between locally compact groups, and that $p : X \to T$ is a G-bundle. Then the action of G on $H \times X$ defined by $s \cdot (h,x) = \big(h\varphi(s^{-1}), s \cdot x\big)$ is free and proper. Furthermore, the action of H on the quotient $\varphi_*(X) = (H \times X)/G$ defined by $h \cdot [k,x] = [hk,x]$ is also free and proper, and the map $\pi : \varphi_*(X) \to T$, defined by $\pi([h,x]) = p(x)$, induces a homeomorphism of $\varphi_*(X)/H$ onto T; that is, $\varphi_*(X)$ is an H-bundle over T.*

Proof. The action of G on $H \times X$ is free and proper because the action on X is, and this implies that $(H \times X)/G$ is a locally compact Hausdorff space. The action of H on $\varphi_*(X)$ is easily seen to be free; to see that it is also proper, fix a compact set K in $\varphi_*(X)$, and suppose $\{h_i\}$ is a net in $\{h \in H : hK \cap K \neq \emptyset\}$. Choose $k_i \in H$, $x_i \in X$ such that $[k_i,x_i] \in K$ and $[h_ik_i,x_i] \in K$. Since the orbit map $q : H \times X \to \varphi_*(X)$ is open, we can choose a compact set L such that $q(L) = K$. Without loss of generality we may suppose $(k_i,x_i) \in L$, and then choose s_i such that $\bigl(h_ik_i\varphi(s_i^{-1}),s_i \cdot x_i\bigr) \in L$; by passing to a subnet we can assume $(k_i,x_i) \to (k,x)$ and $\bigl(h_ik_i\varphi(s_i^{-1}),s_ix_i\bigr) \to (h,\eta)$. The projection L_1 of L on X is compact and the action of G on X is proper, so we can pass to another subnet and assume $s_i \to s$ in G. Thus

$$h_i = h_ik_i\varphi(s_i^{-1})\varphi(s_i)k_i^{-1} \to h\varphi(s)k^{-1},$$

and we have proved that the action of H is proper.

It is easy to verify that the map $\pi : [h,x] \to p(x)$ induces a bijection of $\varphi_*(X)/H$ onto T which will be a homeomorphism if π is continuous and open. We prove first that it is continuous: if $[h_i,x_i] \to [h,x]$, the openness of the quotient map implies there are a subnet $[h_{i_j},x_{i_j}]$ and elements $s_j \in G$ such that $\bigl(h_{i_j}\varphi(s_j^{-1}),s_j \cdot x_{i_j}\bigr) \to (h,x)$, and then $p(x_{i_j}) = p(s_j \cdot x_{i_j}) \to p(x)$. To see that π is open, suppose N is open in $\varphi_*(X)$, and $[h,x] \in N$. Let M_1 and M_2 be neighborhoods of h and x such that $M_1 \times M_2 \subset q^{-1}(N) \subset H \times X$. Then

$$\bigl\{\pi\bigl([k,\eta]\bigr) : k \in M_1,\ \eta \in M_2\bigr\} = \{p(\eta) : \eta \in M_2\}$$

is a neighborhood of $p(x)$ contained in $\pi(N)$, so π is open and the last assertion is established. □

Remark 2.7. If there are local sections $x_i : N_i \to X$ for $p : X \to T$, then X is a principal G-bundle and its isomorphism class is determined by a class $c(X)$ in $\check{H}^1(T,\mathcal{G})$. Since $\eta_i(t) = [e,x_i(t)]$ is a continuous section for $\pi : \varphi_*(X) \to T$, $\varphi_*(X)$ is a principal H-bundle, and it is straightforward to check that its class $c\bigl(\varphi_*(X)\bigr)$ in $\check{H}^1(T,\mathcal{H})$ is simply the image of $c(X)$ under the canonical homomorphism from $\check{H}^1(T,\mathcal{G})$ to $\check{H}^1(T,\mathcal{H})$ induced by φ.

3. Moore cohomology and principal bundles. Throughout this section, G will be a second countable locally compact abelian group and X a second countable locally compact Hausdorff space on which G acts trivially. Recall that a Borel cocycle $w \in Z^2\bigl(G,C(X,\mathbf{T})\bigr)$ is *pointwise trivial*, written $w \in Z^2_{PT}\bigl(G,C(X,\mathbf{T})\bigr)$, if composing with each evaluation map $\varepsilon_x : C(X,\mathbf{T}) \to \mathbf{T}$ gives

a trivial cocycle in $Z^2(G,\mathbf{T})$. We shall construct from each $w \in Z^2_{PT}(G,C(X,\mathbf{T}))$ a $\hat{G}$-bundle E_w over X in such a way that

(1) the class $[E_w]$ depends only on the class $[w]$ in $H^2(G,C(X,\mathbf{T}))$, and
(2) if A is a continuous trace C^*-algebra with spectrum X, and $\alpha : G \to \mathrm{Inn}(A)$ is a pointwise unitary action whose obstruction to being unitary (see [19; Corollary 0.12]) is $[w]$, then E_w is naturally isomorphic to $(A \rtimes_\alpha G)^\wedge$.

The main object of this section is to show that the map $[w] \mapsto [E_w]$ defines an *injective* homomorphism of $H^2_{PT}(G,C(X,\mathbf{T}))$ into $HP(X,\hat{\mathcal{G}})$ (Proposition 3.8). When w is locally trivial, our construction reduces to the one given by Rosenberg in the proof of [21, Theorem 2.5 (b)], and we merely want to observe that the same construction works in the pointwise trivial case and for arbitrary locally compact abelian groups, not just connected ones.

Let $C^n(G,\mathbf{T})$ denote the space of Borel cochains $f : G^n \to \mathbf{T}$, and $\underline{C}^n(G,\mathbf{T})$ the quotient obtained by identifying cochains which agree almost everywhere. As in [9], $\underline{C}^n(G,\mathbf{T})$ has a natural Polish topology for which the coboundary maps $\partial : \underline{C}^n \to \underline{C}^{n+1}$ are continuous [9, Proposition 20]. As in the first paragraph of the proof of [21, Theorem 2.1], w determines a continuous map b_w of X into the quotient $\underline{B}^2(G,\mathbf{T}) = \underline{C}^1(G,\mathbf{T})/\ker \partial = \underline{C}^1(G,\mathbf{T})/\hat{G}$. Now if G is non-discrete, then $\underline{C}^1(G,\mathbf{T})$ is contractible [21, Lemma 2.3]; if G is also compactly generated, so $\hat{G}$ is a Lie group, $\underline{C}^1(G,\mathbf{T})$ is a locally trivial $\hat{G}$-bundle and $\underline{C}^1(G,\mathbf{T}) \to \underline{B}^2(G,\mathbf{T})$ is therefore a universal $\hat{G}$-bundle [21, Proposition 2.4]. The class in $\check{H}^1(X,\hat{\mathcal{G}})$ associated to w in [21, p. 310] is that of the $\hat{G}$-bundle over X pulled back from the universal bundle $\underline{C}^1(G,\mathbf{T})$ along the map b_w. This $\hat{G}$-bundle can be concretely realized as

$$E_w = \{(f,x) \in \underline{C}^1(G,\mathbf{T}) \times X : \partial \bar{f} = b_w(x)\},$$

where $\hat{G}$ acts via $\gamma \cdot (f,x) = (\gamma f,x)$. (Taking $\partial \bar{f}$ instead of ∂f will make our formulas slightly simpler.) Since $\partial : \underline{C}^1 \to \underline{B}^2$ is a universal $\hat{G}$-bundle, the bundle E_w is trivial if and only if b_w lifts to a continuous map of X into $\underline{C}^1$, and Rosenberg argues that such a lifting exists if and only if $[w] = 0$ in $H^2(G,C(X,\mathbf{T}))$ (see the second paragraph of the proof of [21, Theorem 2.1]).

We shall show that for arbitrary locally compact abelian G, the space E_w defined above is a free and proper $\hat{G}$-space with $E_w/\hat{G} = X$, and that its $\hat{G}$-isomorphism class still determines $[w] \in H^2(G,C(X,\mathbf{T}))$. (Notice that $\underline{C}^1(G,\mathbf{T})$ need be neither contractible nor locally trivial in general).

Proposition 3.1. *Suppose G is a locally compact abelian group, X is a locally compact Hausdorff space, and $b : X \to \underline{B}^2(G,\mathbf{T})$ is a continuous map. Then*

$$E_b = \{(f,x) \in \underline{C}^1(G,\mathbf{T}) \times X : \partial\bar{f} = b(x)\}$$

is a locally compact Hausdorff space in the product topology, the action of $\hat{G}$ defined by $\gamma \cdot (f,x) = (\gamma f,x)$ is free and proper, and $r(f,x) = x$ induces a homeomorphism of $E_b/\hat{G}$ onto X. In other words, $r : E_b \to X$ is a $\hat{G}$-bundle.

For the proof we require the following lemma. In the proofs of both the lemma and the proposition we use proper in the sense of [16, Definition 1.2.2]; when Y is locally compact, this is consistent with standard terminology [16, Theorem 1.2.9].

Lemma 3.2. *If G is a locally compact group acting freely and properly on a completely regular space Y, and if Y/G is locally compact, then Y is also locally compact.*

Proof. Let $y \in Y$, and let S be a small closed neighborhood of y. Since $p : Y \to Y/G$ is open and Y/G is locally compact, there is a compact neighborhood M of $p(y)$ such that $M \subset p(S)$. We shall show that $T = S \cap p^{-1}(M)$ is a compact neighborhood of y. It is obviously a neighborhood, so suppose $\{y_\alpha\}_{\alpha\in\Lambda} \subset T$. By passing to a subnet, we may suppose $p(y_\alpha)$ converges to $w \in M$, say. Choose $z \in T$ with $p(z) = w$, and let V be a neighborhood of z which is thin relative to T—in other words, such that $\{s \in G : s \cdot V \cap T \neq \emptyset\}$ is compact in G (this is possible because a subset of a small set is small). Let W be a neighborhood of z. Then $p(W)$ is a neighborhood of $w = p(z)$, and there is an $\alpha_0 \in \Lambda$ so that $\alpha \geq \alpha_0$ implies that $p(y_\alpha) \in p(W)$. Thus we may extract a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that there are $z_\beta \in Y$ converging to z with $p(z_\beta) = p(y_\beta)$. (For example, let Λ' be the collection of pairs (W,α) such that W is a neighborhood of z and $\alpha \in \Lambda$ is such that $p(y_\alpha) \in p(W)$; give Λ' the obvious partial ordering. Then, if $\beta = (W,\alpha) \in \Lambda'$, it suffices to define $y_\beta = y_\alpha$ and to pick z_β in $W \cap p^{-1}(p(y_\alpha))$.) Therefore there are $s_\beta \in G$ such that $s_\beta \cdot z_\beta = y_\beta$. But as each $y_\beta \in T$ and $\{z_\beta\}$ is eventually in V, we may pass to another subnet and suppose that $s_\beta \to s$ in G, and $y_\beta = s_\beta \cdot z_\beta \to s \cdot z$. Thus $\{y_\alpha\}$ has a convergent subnet, and T is compact, as claimed. □

Proof of Proposition 3.1. Theorem 3 of [9] implies that the natural inclusion $\hat{G} \to C^1(G,\mathbf{T})$ induces a continuous bijection of $\hat{G}$ onto the closed subgroup $\ker \partial$ of $\underline{C}^1(G,\mathbf{T})$, and this bijection is therefore a homeomorphism by [9, Proposition 5(b)]. It follows that the map $(f,\gamma f) \to \gamma = (\gamma f)f^{-1}$ is continuous, so that $\underline{C}^1(G,\mathbf{T})$ is a Cartan $\hat{G}$-principal bundle [16, Section 1].

Further, the orbit space is regular [3, Theorem 5.21], and thus $\underline{C}^1(G,\mathbf{T})$ is a proper $\hat{G}$-space by [16, Proposition 1.2.5]. Because ∂ and b are continuous, the space E_b is a closed $\hat{G}$-invariant subset of $\underline{C}^1 \times X$, and as such is itself a proper $\widehat{G}$-space, which is metrizable because $\underline{C}^1$ and X are. It can be routinely verified that, because ∂ is open, p is also open, and hence the bijection of $E_b/\hat{G}$ onto X induced by p is a homeomorphism. In particular, this shows that $E_b/\hat{G}$ is locally compact, and by Lemma 3.2 this in turn implies that E_b is, too. □

Remark 3.3. When $w \in Z^2_{PT}\big(G,C(X,\mathbf{T})\big)$, we'll write E_w in place of E_{b_w}.

Proposition 3.4. *Let* $w \in Z^2_{PT}\big(G,C(X,\mathbf{T})\big)$. *Then* $[w] = 0$ *in* $H^2\big(G,C(X,\mathbf{T})\big)$ *if and only if* E_w *is a trivial* $\hat{G}$*-bundle.*

Proof. If $[w] = 0$, then there is a Borel map $\rho : G \to C(X,\mathbf{T})$ such that $\partial\rho = w$. We define $\psi : X \to C^1(X,\mathbf{T})$ by $\psi(x)(s) = \overline{\rho(s)(x)}$; note that $\psi(x)$ is Borel because it is the composition of ρ with the continuous map ε_x defined by $f \mapsto f(x)$. The continuity of $\rho(s)$ implies that $\psi(x_n) \to \psi(x)$ pointwise, and it follow easily that ψ is continuous from X to $\underline{C}^1(G,\mathbf{T})$—or, strictly speaking, that the composition $\underline{\psi} : X \xrightarrow{\psi} C^1(G,\mathbf{T}) \to \underline{C}^1(G,\mathbf{T})$ is continuous. We clearly have $\partial\big(\underline{\psi}(x)\big) = b_w(x)$, so that $x \mapsto \big(\underline{\psi}(x),x\big)$ is a continuous section for E_w and E_w is trivial.

Reversing this argument is a bit harder: if E_w is trivial, then there is a continuous map $\underline{\psi} : X \to \underline{C}^1(G,\mathbf{T})$ such that $\partial\big(\underline{\psi}(x)\big) = b_w(x)$, but it is not immediately clear how to use $\underline{\psi}$ to define a Borel function $\rho : G \to C(X,\mathbf{T})$. In fact, we do not in general know how to construct elements of $\underline{C}^n\big(G,C(X,\mathbf{T})\big)$ from functions in $C\big(X,\underline{C}^n(G,\mathbf{T})\big)$, and we can only handle the case $n = 1$ here because we know that $\partial(\underline{\psi})$ can be extended to a function w which is a cocycle on all of $G \times G$; the argument we use comes from Rosenberg's proof of [21; Theorem 2.1]. We need two Lemmas.

Lemma 3.5. *Let A be an abelian Polish G-module, and let $\mu \in Z^2(G,A)$ and $f \in C^1(G,A)$ satisfy $\partial f = \mu$ almost everywhere in $G \times G$. Then there is a unique element f_1 of $C^1(G,A)$ such that $f_1 = f$ almost everywhere and $\partial f_1 = \mu$ everywhere on $G \times G$.*

Proof. Since the natural inclusion induces an isomorphism of $H^2 = Z^2/B^2$ onto $\underline{H}^2 = \underline{Z}^2/\underline{B}^2$ [9; Theorem 5], and we are given $[\mu] = 0$ in $\underline{H}^2$, we know that there exists a Borel map $g : G \to A$ such that $\partial g = \mu$ everywhere. Then we have $\partial(f^{-1}g) = 0$ in $\underline{Z}^2$, so that

$$(f^{-1}g)(st) = (f^{-1}g)(s)s \cdot \big((f^{-1}g)(t)\big)$$

for almost all pairs (s,t) in $G \times G$. By [9; Theorem 3], there is a continuous cocycle $\gamma \in Z^1(G,A)$ such that $f^{-1}g = \gamma$ almost everywhere. Then $f_1 = \gamma^{-1}g$ equals f almost everywhere, and satisfies $\partial f_1 = \partial g = \mu$ everywhere.

Suppose that f_2 is another such element of $C^1(G,A)$. Then $\partial f_1 = \partial f_2 = \mu$ implies that $f_1 = \gamma f_2$ for some $\gamma \in Z^1(G,A)$, and $f_1 = \gamma f_2 = f$ almost everywhere implies that $\gamma = 1$ almost everywhere; thus $\gamma = 1$ and $f_1 = f_2$. □

Lemma 3.6. *Suppose that $w \in Z^2_{PT}(G,C(X,\mathbf{T}))$, and that we have a sequence (f_n,x_n) in E_w which converges to (f,x). Let g_n, g be elements of $C^1(G,\mathbf{T})$ which represent f_n, f in $\underline{C}^1(G,\mathbf{T})$ and which satisfy $\partial g_n = (\varepsilon_{x_n})_*(w)$ and $\partial g = (\varepsilon_x)_*(w)$ everywhere. Then $g_n(s)$ converges to $g(s)$ for all $s \in G$.*

Proof. Suppose there exists $s \in G$ such that $g_n(s)$ does not converge to $g(s)$. By passing to a subsequence, we may suppose that

$$|g_n(s) - g(s)| \geq \varepsilon \qquad \text{for all } n, \tag{3.1}$$

which ensures that no subsequence of $\{g_n(s)\}$ can converge to $g(s)$. We know that $f_n \to f$ in $\underline{C}^1(G,\mathbf{T})$, and hence by [9; Proposition 6] there is a subsequence $\{f_{n_j}\}$ converging almost everywhere to f. Since $g_{n_j} = f_{n_j}$ almost everywhere, there is a Borel null-set L such that

$$g_{n_j}(t) \to g(t) \qquad \text{for all } t \in G \setminus L. \tag{3.2}$$

Since w takes values in $C(X,\mathbf{T})$, $g_{n_j}(st)g_{n_j}(s)^{-1}g_{n_j}(t)^{-1} = w(s,t)(x_{n_j})$ converges to $w(s,t)(x) = g(st)g(s)^{-1}g(t)^{-1}$, which, in view of (3.1) and (3.2), implies that $\{g_{n_j}(st)\}$ cannot converge to $g(st)$ for our fixed s and any $t \in G \setminus L$. Therefore, $t \in G \setminus L$ implies that $st \in L$, and we have $G = L \cup s^{-1} \cdot L$, which is impossible since L has Haar measure zero. □

End of the proof of Proposition 3.4. As before, if E_w is trivial, then there is a continuous map $\underline{\psi} : X \to \underline{C}^1(G,\mathbf{T})$ such that $\partial \circ \underline{\psi} = b_w$. By Lemma 3.5, for each $x \in X$ there is a unique element $\psi(x)$ of $C^1(X,\mathbf{T})$ such that $\psi(x)$ represents $\underline{\psi}(x)$, and such that $\partial(\psi(x)) = w(\cdot,\cdot)(x)$ everywhere. Lemma 3.6 implies that if $x_n \to x$, then $\psi(x_n) \to \psi(x)$ pointwise. Thus we can define $\rho : G \to C(X,\mathbf{T})$ by $\rho(s)(x) = \psi(x)(s)$. Then formally we have

$$[\partial\rho(s,t)](x) = \partial(\psi(x))(s,t) = w(s,t)(x),$$

but we still have to show that ρ is Borel.

Since the Borel structure on $C(X,\mathbf{T})$ is generated by the (compact-open) topology, it will be enough to prove that for each $g \in C(X,\mathbf{T})$, K compact in X, and $\varepsilon > 0$, the set

$$\begin{aligned} E &= \rho^{-1}\big(\{f \in C(X,\mathbf{T}) : |f(x) - g(x)| \le \varepsilon \text{ for all } x \in K\}\big) \\ &= \{s \in G : |\psi(x)(s) - g(x)| \le \varepsilon \text{ for all } x \in K\} \end{aligned}$$

is Borel in G. Let $\{x_n\}$ be a countable dense set in K. Since for each fixed x, the function $\psi(x)$ is Borel,

$$E_n = \{s \in G : |\psi(x_n)(s) - g(x_n)| \le \varepsilon\}$$

is Borel, and hence so is $\bigcap_{n=1}^{\infty} E_n$. But both $\psi(\cdot)(s)$ and g are continuous, so $E = \bigcap_{n=1}^{\infty} E_n$, and E is Borel too. □

Remark 3.7. We have actually shown that every global section σ of E_w has the form

$$\sigma(x) = \big(\rho(\,\cdot\,)(x), x\big)$$

for some $\rho \in C^1\big(G, C(X,\mathbf{T})\big)$ such that $\partial\bar{\rho} = w$.

Proposition 3.8. *The map $w \mapsto E_w$ constructed in Proposition 3.1 induces an injective homomorphism of $H^2_{PT}\big(G, C(X,\mathbf{T})\big)$ into $HP(X,\mathcal{G})$.*

Proof. If b and c are continuous maps of X into $\underline{B}^2(G,\mathbf{T})$, then it can be routinely verified that the map $\big((f,x),(g,x)\big) \mapsto (fg,x)$ induces a continuous bijection of $E_b * E_c = (E_b \underset{X}{\times} E_c)/\hat{G}$ onto E_{bc}, and by Remark 2.3 this must be an isomorphism of $\hat{G}$-bundles. Thus, in particular, $[E_{w\tau}] = [E_w][E_\tau]$ in $HP(X,\mathcal{G})$. If $[w] = [\tau]$ in $H^2\big(G, C(X,\mathbf{T})\big)$, then $[w\bar{\tau}] = 0$, and hence by Proposition 3.4, $[E_w] = [E_{w\bar{\tau}}][E_\tau]$. Therefore the map $[w] \mapsto [E_w]$ is a well-defined homomorphism, which is injective by Proposition 3.4. □

We shall now start the process of identifying the range of the homomorphism $w \mapsto E_w$. We shall show that, if $\psi_s : \hat{G} \to \mathbf{T}$ denotes evaluation at $s \in G$, then the $\mathbf{T}$-bundle $(\psi_s)_*(E_w)$ (see Proposition 2.6) is trivial for all $s \in G$. In fact, this property characterizes the $\hat{G}$-bundles of the form E_w, and we shall later give two proofs of this: the first, in Section 4, uses operator algebras, whereas the second, which is a special case of the argument in Section 8, is direct but harder.

Proposition 3.9. *Suppose that $\varphi : H \to G$ is a continuous homomorphism of locally compact abelian groups. Then the induced homomorphism $\varphi^* : H^2\big(G, C(X,\mathbf{T})\big) \to H^2\big(H, C(X,\mathbf{T})\big)$ carries $H^2_{PT}\big(G, C(X,\mathbf{T})\big)$ into $H^2_{PT}\big(H, C(X,\mathbf{T})\big)$, and the bundle $E_{\varphi^*(w)}$ is naturally isomorphic to $\hat{\varphi}_*(E_w)$.*

Proof. The induced homomorphism φ^* is defined by

$$\varphi^*(w)(h,k) = w\bigl(\varphi(h),\varphi(k)\bigr)$$

for $w \in Z^2\bigl(G,C(X,\mathbf{T})\bigr)$, so if $\rho : G \to \mathbf{T}$ satisfies $\partial\rho = (\varepsilon_x)_*(w)$, then $\partial(\rho\circ\varphi) = (\varepsilon_x)_*\bigl(\varphi^*(w)\bigr)$, and $\varphi^*(w)$ is pointwise trivial if w is. We define a map $\Phi : \hat{H}\times E_w \to E_{\varphi^*(w)}$ by $\Phi(\gamma,f,x) = \bigl(\gamma(g\circ\varphi),x\bigr)$, where g is the unique function in $C^1(G,\mathbf{T})$ representing $f \in \underline{C}^1(G,\mathbf{T})$ and satisfying $\partial g = (\varepsilon_x)_*(w)$ (see Lemma 3.5): since γ is a homomorphism,

$$\begin{aligned}\partial\bigl(\gamma(g\circ\varphi)\bigr) &= \partial(g\circ\varphi) = \partial g\circ(\varphi\times\varphi)\\ &= (\varepsilon_x)_*(w)\circ(\varphi\times\varphi) = (\varepsilon_x)_*(w\circ\varphi\times\varphi) = (\varepsilon_x)_*\bigl(\varphi^*(w)\bigr),\end{aligned}$$

so Φ does take values in $E_{\varphi^*(w)}$. To see that Φ is continuous, we suppose that $(\gamma_n,f_n,x_n) \to (\gamma,f,x)$. Lemma 3.6 implies that $g_n \to g$ pointwise on G, and hence $\gamma(g_n\circ\varphi) \to \gamma(g\circ\varphi)$ pointwise on H. By [9; Proposition 6], this implies that $\gamma(g_n\circ\varphi) \to \gamma(g\circ\varphi)$ in $\underline{C}^1(H,\mathbf{T})$, and hence $\bigl(\gamma(g_n\circ\varphi),x_n\bigr) \to \bigl(\gamma(g\circ\varphi),x\bigr)$ in $E_{\varphi^*(w)}$.

The space $\hat{\varphi}_*(E_w)$ is by definition the orbit space for the action of $\hat{G}$ on $\hat{H}\times E_w$ given by

$$\chi\cdot(\gamma,f,x) = \bigl(\gamma\bigl(\hat{\varphi}(\chi)^{-1}\bigr),\chi f,x\bigr) = \bigl(\gamma(\overline{\chi\circ\varphi}),\chi f,x\bigr);$$

if g is the unique lifting of f to $C^1(G,\mathbf{T})$, then χg is the unique lifting of χf, and hence Φ is constant on $\hat{G}$-orbits. We now have a continuous $\widehat{H}$-equivariant map Φ' of $\hat{\varphi}_*(E_w) = (\hat{H}\times E_w)/\hat{G}$ into $E_{\varphi^*(w)}$, which clearly respects the bundle projections onto X, and is therefore a bundle isomorphism by Remark 2.3. □

Corollary 3.10. *For $s \in G$, define $\psi_s : \hat{G} \to \mathbf{T}$ by $\psi_s(\gamma) = \gamma(s)$. If $w \in Z^2_{PT}\bigl(G,C(X,\mathbf{T})\bigr)$, then $(\psi_s)_*(E_w)$ is a trivial $\mathbf{T}$-bundle for all $s \in G$.*

Proof. The map ψ_s is $\hat{\varphi}_s$, where $\varphi_s : \mathbf{Z} \to G$ is defined by $\varphi_s(n) = s^n$. Thus the proposition implies that $(\psi_s)_*(E_w) = E_{\varphi_s^*(w)}$. But $H^2\bigl(\mathbf{Z},C(X,\mathbf{T})\bigr) = 0$, so $E_{\varphi_s^*(w)}$ must be trivial by Proposition 3.4. □

Remark 3.11. The class of $\hat{G}$-bundles E over X satisfying

(3.3) $$(\psi_s)_*(E) \cong X\times\mathbf{T} \qquad \text{for all } s \in G$$

has occurred before in work of Smith [22, 23]. He calls a $\hat{G}$-bundle $p : E \to X$ *characteristic* if there is a function $f : G \times E \to \mathbf{T}$, which is Borel on G and continuous on E, and which satisfies

$$f(s,\gamma \cdot \xi) = \gamma(s) f(s,\xi) \tag{3.4}$$

for $s \in G$, $\xi \in E$, and $\gamma \in \hat{G}$. It is easy to check that, given such an f and $s \in G$, the map $\varphi : (\psi_s)_*(E) \to X \times \mathbf{T}$ defined by

$$\varphi([z,\xi]) = (p(\xi), f(s,\xi), z) \tag{3.5}$$

is a bundle isomorphism, and hence his characteristic bundles satisfy our criterion (3.3). On the other hand, it follows from Theorem 4.4 (or from Section 8) that (3.3) implies $E \cong E_w$ for some $w \in Z^2_{PT}(G, C(X,\mathbf{T}))$, and then we can set $f(s,(g,x)) = g_1(s)$, where g_1 is the unique representative of $g \in \underline{C}^1(G,\mathbf{T})$ such that $\partial g_1 = (\varepsilon_x)_*(w)$ everywhere. (Lemma 3.6 implies that $f(s,\cdot)$ is continuous on E_w for each fixed s, and the argument used to prove Proposition 3.4 shows that $s \mapsto f(s,\cdot)$ is Borel as a map from G to $C(E,\mathbf{T})$.) Therefore a bundle satisfies criterion (3.3) if and only if it is characteristic. It is mildly interesting to note that, if for each $s \in G$ we can find $f(s,\cdot)$ satisfying (3.4), then (3.5) defines a trivialization of $(\psi_s)_*(E)$, and E is characteristic; thus the measurability requirement on f appears to be unnecessary.

Smith was interested in those principal $\hat{G}$-bundles over X which could arise as the spectrum of *commutative* twisted crossed products $A \rtimes_{\alpha,u} G$ [22]. (Here we use the C^*-algebraic twisted crossed products of [15], rather than those based on $L^1(G,A)$ which Smith was using, but his results would apply to these ones too.) For such a crossed product to be commutative, A must be commutative, say $A = C_0(X)$, G must be abelian and act trivially on X, and $u \in Z^2(G, C(X,\mathbf{T}))$ must be symmetric; since Kleppner has shown that for G abelian a cocycle in $Z^2(G,\mathbf{T})$ is symmetric if and only if it is trivial in $H^2(G,\mathbf{T})$ [7], and since symmetry of an element of $Z^2(G, C(X,\mathbf{T}))$ is a pointwise property, u is symmetric if and only if it is pointwise trivial in our sense. Hence Smith was studying the spectrum of the twisted crossed products $C_0(X) \rtimes_{\mathrm{id},w} G$ for $w \in Z^2_{PT}(G, C(X,\mathbf{T}))$. He showed that it was always a principal $\hat{G}$-bundle over X, and that the $\hat{G}$-bundles which arose this way were precisely the characteristic ones.

Smith's results are of course compatible with ours. Indeed, by Theorem 3.7 of [15], there is an action β of G on $C_0(X,\mathcal{K})$ by inner automorphisms such that $c(\beta) = [w]$ and

$$C_0(X) \rtimes_{\mathrm{id},w} G \cong C_0(X,\mathcal{K}) \rtimes_\beta G.$$

Since w is pointwise trivial, β is pointwise unitary, and the spectrum $(C_0(X,\mathcal{K}) \rtimes_\beta G)^\wedge$ will be a $\hat{G}$-bundle which is isomorphic to E_w by Proposition 4.1. Thus characterizing the bundles of the form $(C_0(X) \rtimes_{\mathrm{id},w} G)^\wedge$ amounts to identifying the range of the map $w \mapsto [E_w]$. □

4. Pointwise unitary actions. Suppose now that $\alpha : G \to \operatorname{Aut}(A)$ is a pointwise unitary action on a continuous-trace algebra consisting of inner automorphisms. We want to show that the map $[w] \mapsto [E_w]$ carries $c(\alpha) \in H^2_{PT}(G,C(X,\mathbf{T}))$, the algebraic obstruction to lifting $\alpha : G \to \operatorname{Inn}(A)$ to a homomorphism $u : G \to \mathcal{UM}(A)$ [19; Theorem 0.11], into the class $\zeta(\alpha)$ of the $\hat{G}$-bundle $\operatorname{Res} : (A \rtimes_\alpha G)^\wedge \to X$ defined by restriction of irreducible representations to $A \subseteq \mathcal{M}(A \rtimes_\alpha G)$ [13; Theorem 1.7]. We shall then use this to complete our identification of the image of the homomorphism $[w] \mapsto [E_w]$, and discuss the relationship of our results to Rosenberg's.

Proposition 4.1. *Suppose A is a separable continuous-trace C^*-algebra with spectrum X, $\alpha : G \to \operatorname{Inn} A$ is a pointwise unitary action of an abelian group as inner automorphisms, and $u : G \to \mathcal{UM}(A)$ is a Borel map implementing α, so that $c(\alpha) \in H^2(G,C(X,\mathbf{T}))$ is represented by the cocycle $w \in Z^2(G,C(X,\mathbf{T}))$ satisfying $u_s u_t = w(s,t)u_{st}$. Then the map $h : (f,\pi) \mapsto \pi \times f\pi(u)$ is a $\hat{G}$-equivariant homeomorphism of E_w onto $((A \rtimes_\alpha G))^\wedge$ such that*

$$\begin{tikzcd} E_w \arrow[rr, "h"] \arrow[dr, "p"'] & & ((A \rtimes_\alpha G))^\wedge \arrow[dl, "\operatorname{Res}"] \\ & X & \end{tikzcd}$$

Proof. We view A as the algebra $\Gamma_0(E)$ of sections of a C^*-bundle E over X, so that $\hat{A}$ can be naturally identified with the set of evaluation maps $\varepsilon_x : a \mapsto a(x)$. Next we observe that $h(f,x)$ is a well-defined representation of $A \rtimes_\alpha G$: if $(f,x) \in E_w$, then $s \mapsto f(s)u_s(x)$ is almost everywhere a Borel homomorphism, and hence is equal almost everywhere to a continuous homomorphism $U : G \to \mathcal{UM}(A_x)$ such that (ε_x,U) is a covariant representation of (A,G,α), which we shall denote by $(\varepsilon_x,fu(x))$. By [18, Proposition 2.1], every element of $\operatorname{Res}^{-1}(x)$ then has the form

$$\gamma \cdot \bigl(\varepsilon_x \times fu(x)\bigr) = \varepsilon_x \times \bigl(\gamma fu(x)\bigr) = h(\gamma f,x) = h\bigl(\gamma \cdot (f,x)\bigr),$$

so h is $\hat{G}$-equivariant and surjective and two such elements $h(f_i,x)$ agree if and only if $f_i = f_j$ almost everywhere, i.e., if and only if $f_i = f_j$ in $\underline{C}^1(G,\mathbf{T})$. Since the diagram trivially commutes and h is a bijection, it will automatically be a homeomorphism if it is continuous ([13, Theorem 1.7] and Remark 2.3). Suppose $(f_n,x_n) \to (f,x)$ in E_w. Then $x_n \to x$, and by passing to a subsequence we may assume $f_n \to f$ almost everywhere in G [9, Proposition 6]. To establish that

$\pi_n = h(f_n,x_n)$ converges to $\pi = h(f,x)$, we have to show that for each $\varepsilon > 0$, unit vector $\eta \in H(\pi)$ and $z \in C_c(G,A) \subset A \rtimes_\alpha G$, there exists N such that if $n \geq N$, we can find a unit vector $\xi \in H(\pi_n)$ satisfying

$$\left|\big(\pi_n(z)\xi \mid \xi\big) - \big(\pi(z)\eta \mid \eta\big)\right| < \varepsilon.$$

Since the problem is local in X, we may as well suppose that A is the C^*-algebra defined by a Hilbert bundle H over X. Then $H(\pi_n) = H_{x_n}$, $H(\pi) = H_x$, and each unit vector in H_x has the form $\eta(x)$ for some $\eta \in \Gamma(H)$: by localizing and normalizing, we may suppose $\|\eta(x_n)\| = 1$ for all n. For any $a \in A$, the function $y \mapsto \big(a(y)\eta(y) \mid \eta(y)\big)$ is continuous, and each $z(s)u_s$ is in A, so

$$f_n(s)\big(z(s)(x_n)u_s(x_n)\eta(x_n) \mid \eta(x_n)\big) \to f(s)\big(z(s)(x)u_s(x)\eta(x) \mid \eta(x)\big)$$

for almost all $s \in G$. Further, we have

$$\left|f_n(s)\big(z(s)(x_n)u_s(x_n)\eta(x_n) \mid \eta(x_n)\big)\right| \leq \|z(s)\|,$$

and the dominated convergence theorem implies

$$\begin{aligned}
\big(\pi_n(z)\eta(x_n) \mid \eta(x_n)\big) &= \int \big(z(s)(x_n)f_n(s)u_s(x_n)\eta(x_n) \mid \eta(x_n)\big)\,ds \\
&\to \int \big(z(s)(x)f(s)u_s(x)\eta(x) \mid \eta(x)\big)\,ds \\
&= \big(\pi(z)\eta(x) \mid \eta(x)\big).
\end{aligned}$$

Thus $\pi_n \to \pi$ in $((A \rtimes_\alpha G))^\wedge$, h is continuous, and the result follows. □

Lemma 4.2. *Suppose $\varphi : G \to H$ is a continuous homomorphism between two locally compact abelian groups, A is a separable continuous-trace algebra and $\alpha : H \to \operatorname{Aut} A$ is pointwise unitary. Then the map*

$$\vartheta : (\gamma,\pi \times U) \mapsto \pi \times \gamma(U \circ \varphi)$$

induces a $\hat{G}$-equivariant homeomorphism h of $\hat{\varphi}_*\big(((A \rtimes_\alpha H))^\wedge\big)$ onto $((A \rtimes_{\alpha\circ\varphi} G))^\wedge$ such that

$$\begin{tikzcd}
\hat{\varphi}_*\big(((A \rtimes_\alpha H))^\wedge\big) \arrow[rr, "h"] \arrow[dr, "\hat{\varphi}_*(\mathrm{Res})"'] & & ((A \rtimes_{\alpha\circ\varphi} G))^\wedge \arrow[dl, "\mathrm{Res}"] \\
& \hat{A} &
\end{tikzcd}$$

commutes.

Proof. It follows easily from [18, Proposition 2.1] that ϑ is a well-defined map of $((A \rtimes_\alpha H))^\wedge$ onto $((A \rtimes_{\alpha\circ\varphi} G))^\wedge$. To see that ϑ is continuous, first observe that it is the composition

$$\hat{G} \times ((A \rtimes H))^\wedge \xrightarrow{\mathrm{id}\times\hat{\varphi}} \hat{G} \times ((A \rtimes_{\alpha\circ\varphi} G))^\wedge \xrightarrow{\text{dual action}} ((A \rtimes_{\alpha\circ\varphi} G))^\wedge,$$

where $\hat{\varphi}$ sends $\pi \times U$ to $\pi \times (U \circ \varphi)$. The homomorphism φ induces a homomorphism $\mathrm{id} \times \varphi : A \rtimes_{\alpha\circ\varphi} G \to \mathcal{M}(A \rtimes_\alpha H)$, and it in turn induces a continuous map $(\mathrm{id} \times \varphi)^*$ from the space $\mathcal{I}(A \rtimes H)$ of closed ideals to $\mathcal{I}(A \rtimes G)$ [1, Proposition 9]: if $\pi \times U \in ((A \rtimes_\alpha H))^\wedge$, then

$$\begin{aligned}(\mathrm{id} \times \varphi)^*\big(\ker(\pi \times U)\big) &= \{z \in A \times G : (\mathrm{id} \times \varphi)(z) \cdot (A \times H) \in \ker \pi \times U\} \\ &= \big\{z \in A \times G : \pi \times U\big(\mathrm{id} \times \varphi(z)\big) = 0\big\} \\ &= \ker\big(\pi \times (U \circ \varphi)\big).\end{aligned}$$

Since both $\pi \times U$, $\pi \times (U \circ \varphi)$ are irreducible, and $A \rtimes_{\alpha\circ\varphi} G$, $A \rtimes_\alpha H$ are type I (they have Hausdorff spectrum by [13, Proposition 1.5]), this implies that $\hat{\varphi} : \pi \times U \to \pi \times (U \circ \varphi)$ is continuous. It follows that ϑ is continuous.

It is easy to check that ϑ is constant on $\hat{H}$-cosets, and hence induces a continuous map h of $\big(\hat{G} \times (A \rtimes_\alpha H)^\wedge\big)/\hat{H}$ onto $((A \rtimes_{\alpha\circ\varphi} G))^\wedge$. We trivially have that $\mathrm{Res} \circ h = p$ and h is $\hat{G}$-equivariant. Since both actions of $\hat{G}$ are free and proper ([13, Theorem 1.7] and our Proposition 3.1), it follows from Remark 2.3 that h is a homeomorphism. □

Corollary 4.3. *Suppose G is a locally compact abelian group, A is a continuous-trace C^*-algebra and $\alpha : G \to \mathrm{Aut}\, A$ is pointwise unitary. For $s \in G$, we define $\varphi_s : \mathbf{Z} \to G$ by $\varphi_s(n) = s^n$. Then α_s is inner if and only if the $\mathbf{T}$-bundle $\hat{\varphi}_s\big(((A \rtimes_\alpha G))^\wedge\big)$ is trivial.*

Proof. The automorphism α_s is inner if and only if the action $\alpha \circ \varphi_s$ of $\mathbf{Z}$ is unitary, which happens if and only if $((A \rtimes_{\alpha\circ\varphi} \mathbf{Z}))^\wedge$ is a trivial $\mathbf{T}$-bundle [18, Proposition 2.5]. Therefore the result follows from the proposition. □

Theorem 4.4. *Let G be a second countable locally compact abelian group acting trivially on a second countable locally compact space X, and for $s \in G$ define $\psi_s : \hat{G} \to \mathbf{T}$ by $\psi_s(\gamma) = \gamma(s)$. Then the map $w \mapsto E_w$ of Proposition 3.1 induces an isomorphism of $H^2_{PT}\big(G, C(X, \mathbf{T})\big)$ onto*

$$\Big\{[E] \in HP(X, \hat{\mathcal{G}}) : (\psi_s)_*(E) \textit{ is trivial for all } s \in G\Big\}.$$

Proof. After the results of Section 3, it only remains to show the surjectivity. So suppose E is a $\hat{G}$-bundle over X such that $(\psi_s)_*(E)$ is a trivial $\mathbf{T}$-bundle for all $s \in G$. By [13, Proposition 1.13] there is a pointwise unitary action α of G on $A = C_0(X,\mathcal{K})$ such that $((A \rtimes_\alpha G))^\wedge$ is $\hat{G}$-isomorphic to E. By Lemma 4.2, we have

$$(\psi_s)_*(E) = (\hat{\varphi}_s)_*(E) \cong ((A \rtimes_{\alpha\circ\varphi_s} \mathbf{Z}))^\wedge,$$

and because this is trivial, $\alpha_s = \alpha \circ \varphi_s(1)$ is inner (Corollary 4.3). Thus $\alpha : G \to \operatorname{Inn} A$, and is implemented by a Borel map $u : G \to \mathcal{UM}(A)$. If $u_s u_t = w(s,t)u_{st}$, then $w \in Z^2(G,C(X,\mathbf{T}))$ is pointwise trivial because α is pointwise unitary, and Proposition 4.1 implies $E_w \cong ((A \rtimes_\alpha G))^\wedge \cong E$, as required. □

As we mentioned in the introduction, our results have been motivated by a theorem of Rosenberg concerning connected groups [21; Theorem 2.5], and this can be deduced from our Proposition 4.1 using duality.

Theorem 4.5 (Rosenberg). *Let G be a second countable connected locally compact abelian group, and X a second countable locally compact space. Suppose that $\check{H}^2(X;\mathbf{Z})$ is countable, and that G acts trivially on X. Then the map $w \mapsto E_w$ induces an isomorphism of $H^2_{PT}(G,C(X,\mathbf{T}))$ onto $\check{H}^1(X,\hat{\mathcal{G}})$.*

Proof. Since G is connected, it is compactly generated, $\hat{G}$ is a Lie group, and all $\hat{G}$-bundles are locally trivial [16; Theorem 4.1]; hence $\check{H}^1(X,\hat{\mathcal{G}}) = HP(X,\hat{\mathcal{G}})$. Thus we only have to prove that every principal $\hat{G}$-bundle E has the form E_w. But if E is a $\hat{G}$-bundle, the dual action of G on $A = C_0(E) \rtimes \hat{G}$ is locally unitary [19; Proposition 3.1], and $\alpha(G) \subseteq \operatorname{Inn}(A)$ because G is connected and $\operatorname{Inn}(A)$ is open in $\operatorname{Aut}_{C_0(X)}(A)$ [19; Theorem 0.8]. If $w \in Z^2(G,C(X,\mathbf{T}))$ represents $c(\alpha)$, then w is pointwise trivial because α is pointwise unitary, and $E_w \cong (A \rtimes_\alpha G)^\wedge$ by Proposition 4.1. Since $(A \rtimes_\alpha G)^\wedge \cong E$ by [18; Proposition 3.1], this proves the result. □

This proof of Rosenberg's theorem does use some fairly heavy tools from operator algebras, and therefore seems to be substantially more complicated than the original. However, Rosenberg agrees with us that the argument in [21] may be inadequate as it stands, and the alternative direct proof of Theorem 4.5 which he has shown us also relies on some sophisticated machinery. The potential problem in [21] occurs at the end of the proof of part (b), where it is asserted that a continuous function $f : X \to \underline{B}^2(G,\mathbf{T})$ gives an element of $\underline{Z}^2(G,C(X,\mathbf{T}))$ (see the comments concerning Lemma 3.5 above); for this to be true one will certainly need some hypotheses on G, since we give an example below (Example 4.7) where G is non-discrete, so $\underline{C}^1(G,\mathbf{T})$ is still contractible, but $w \mapsto [E_w]$ is not surjective.

Putting our theorem together with Rosenberg's gives an amusing topological corollary:

Corollary 4.6. *If G is connected and E is a $\hat{G}$-bundle with $\check{H}^2(E/\hat{G},\mathbf{Z})$ countable, then $(\psi_s)_*(E)$ is a trivial $\mathbf{T}$-bundle for all $s \in G$.*

Of course, the above proof of this purely topological fact must be unnecessarily convoluted, and at least for compact X a simple direct argument can be given. For if E has transition functions $\lambda_{ij} : N_{ij} \to \hat{G}$ and X is compact, we may suppose the cover is finite and the ranges of the λ_{ij} all lie in a compact subset K of $\hat{G}$. Then $\psi_t \to \psi_s$ uniformly on K as $t \to s$, and there is a neighborhood N of s such that $t \in N$ implies that $|\lambda_{ij}(x)(t) - \lambda_{ij}(x)(s)| < \sqrt{2}$ for all i, j and $x \in N_{ij}$. Then applying the principal branch of log to $\lambda_{ij}(\,\cdot\,)(t)\lambda_{ij}(\,\cdot\,)(s)^{-1}$ gives a cocycle with values in $\mathbf{R}$, which has the form $\partial\rho$ since $\check{H}^1(X,\mathcal{R}) = 0$, and we have $\partial(\exp\rho)\{\psi_s \circ \lambda_{ij}\} = \{\psi_t \circ \lambda_{ij}\}$. Since G is connected and $(\psi_e)_*(E)$ is the trivial bundle, this proves that $(\psi_s)_*(E)$ is trivial for all s.

In the event that X is paracompact, but not necessarily compact, the referee has suggested the following argument. Since G is connected and abelian, general structure theory implies that $\hat{G} \cong \hat{K} \times \mathbf{R}^n$ where $\hat{K}$ is discrete and torsion-free [3; Theorems 24.25 and 24.30]. Since $\mathbf{R}^n$ is contractable, in order to show that

$$(\psi_s)_* : \check{H}^1(X,\mathcal{G}) \to \check{H}^1(X,\mathcal{S}) \cong \check{H}^2(X;\mathbf{Z})$$

is trivial, it is enough to show that $(\psi_k)_* : \check{H}^1(X;\hat{K}) \to \check{H}^1(X,\mathcal{S})$ is trivial. Since $\hat{K}$ is torsion-free, we can, with suitable hypotheses on X and K, apply the universal coefficient theorem [24; Theorem 5.5.10] to conclude that

$$\check{H}^1(X;\hat{K}) \cong \check{H}^1(X;\mathbf{Z}) \otimes \hat{K}, \tag{4.1}$$

and then $(\psi_k)_*$ corresponds to pairing with k. Thus the image ends up in the image of $\check{H}^1(X;\mathbf{Z}) \otimes \mathbf{T}$ in $\check{H}^1(X;\mathbf{T})$, which in turn maps to $\check{H}^1(X,\mathcal{S})$. But a straightforward calculation with cocycles shows that the map from $\check{H}^1(X;\mathbf{T})$ to $\check{H}^1(X,\mathcal{S})$ is zero on $\check{H}^1(X;\mathbf{Z}) \otimes \mathbf{T}$. Since (4.1) is valid whenever either $\hat{K}$ is finitely generated or the homology of X is finitely generated, the above gives a topological proof of Corollary 4.6 in these cases. Note that X having finitely generated homology actually implies that $\check{H}^2(X;\mathbf{Z})$ is finitely generated [24; Theorem 5.5.3]—hence countable.

Example 4.7. Let $G = \mathbf{R} \times \mathbf{Z}$. The map $(r,s) \to \big(r, \exp(2\pi i s)\big)$ of $\mathbf{R}^2$ onto $\mathbf{R} \times \mathbf{T}$ induces (via the long exact sequence of sheaf cohomology) an isomorphism of $\check{H}^1(X,\hat{\mathcal{G}})$ onto $\check{H}^2(X,\mathbf{Z})$, and the evaluation map $\psi_{(0,1)} : (r,z) \to z$ induces an isomorphism of $\check{H}^1(X,\hat{\mathcal{G}})$ onto $\check{H}^1(X,\mathcal{S})$. Thus there are no non-trivial locally trivial $\hat{G}$-bundles E such that $(\psi_s)_*(E)$ is trivial for all $s \in G$. This is consistent

with our theorem, since $H^2_{PT}(G,C(X,\mathbf{T})) = 0$ for any space X. To see this, we recall from [19, Theorem 4.1] that

$$H^2(\mathbf{R},C(X,\mathbf{T})) = H^3(\mathbf{R},C(X,\mathbf{T})) = 0 = H^2(\mathbf{Z},C(X,\mathbf{T})).$$

We can therefore deduce from the Lyndon-Hochschild-Serre spectral sequence for Moore cohomology that

$$H^2(\mathbf{R}\times\mathbf{Z},C(X,\mathbf{T})) \cong H^1(\mathbf{R},H^1(\mathbf{Z},C(X,\mathbf{T})));$$

indeed, a specific isomorphism is given by sending $\varphi \in Z^1(\mathbf{R},H^1(\mathbf{Z},C(X,\mathbf{T})))$ to the cocycle $\mu_\varphi \in Z^2(G,C(X,\mathbf{T}))$ given by

$$\mu_\varphi((s,n),(t,m)) = \varphi(t)(n)$$

(see, for example, [14, Appendix 2]). Now

$$H^1(\mathbf{R},H^1(\mathbf{Z},C(X,\mathbf{T}))) \cong \mathrm{Hom}(\mathbf{R},C(X,\mathbf{T})) \cong C(X,\hat{\mathbf{R}}),$$

and this isomorphism is functorial in X, so we have a commutative diagram

$$\begin{tikzcd}
H^2(\mathbf{R}\times\mathbf{Z},C(X,\mathbf{T})) \arrow[r] \arrow[d, "(\varepsilon_x)_*"'] & C(X,\hat{\mathbf{R}}) \arrow[d, "\varepsilon_x"] \\
H^2(\mathbf{R}\times\mathbf{Z},\mathbf{T}) \arrow[r] & \hat{\mathbf{R}}
\end{tikzcd}$$

In particular, $(\varepsilon_x)_*(\mu) = 0$ for all x if and only if the corresponding function vanishes identically, and therefore the pointwise trivial part of $H^2(\mathbf{R}\times\mathbf{Z},C(X,\mathbf{T}))$ is 0, as claimed. □

5. The Λ-invariant. We are interested in the Moore groups $H^2(G,\cdot\,)$ because they contain the obstruction to implementing an automorphism group $\alpha : G \to \mathrm{Inn}(A)$ by a unitary group $u : G \to \mathcal{UM}(A)$. When this happens, of course, the system is easy to analyze; for example, the crossed product $A \rtimes_\alpha G$ is isomorphic to $A \otimes_{\max} C^*(G)$. Even if α does not consist of inner automorphisms we can try to implement $\alpha|_N$ whenever N is a normal subgroup of G such that $\alpha(N) \subseteq \mathrm{Inn}(A)$. However, in order to obtain useful information about $A \rtimes_\alpha G$, we have to know also that the resulting homomorphism $u : N \to \mathcal{UM}(A)$ is compatible with the action of all of G; specifically, we require that

$$\alpha_s(u_n) = u_{sns^{-1}} \qquad \text{for } n \in N, \text{ and } s \in G. \tag{5.1}$$

We shall call a strictly continuous homomorphism $u : N \to \mathcal{UM}(A)$ satisfying $\alpha|_N = \mathrm{Ad}\, u$ and (5.1) a *Green twisting map* for α on N, for reasons we shall shortly explain.

When α has a Green twisting map u on N, we can form the *restricted crossed product* $A \rtimes^u_{\alpha,N} G$, which is the quotient of $A \rtimes_\alpha G$ whose representations are given by covariant pairs (π,U) satisfying $\pi(u_n) = U_n$ for all $n \in N$. These crossed products were introduced and used by Green in his version of the Mackey machine for crossed products [1] (see also [11]). They behave very much like ordinary crossed products by G/N, and it is often possible to obtain information about $A \rtimes_\alpha G$ from $A \rtimes^u_{\alpha,N} G$. Indeed, Olesen and Pedersen [12; Theorem 2.4] showed that when G is abelian we can recover $A \rtimes_\alpha G$ as the induced C^*-algebra $\mathrm{Ind}_{N^\perp}^{\hat{G}}(A \rtimes^u_{\alpha,N} G, \hat{\alpha})$ consisting of those functions $f \in C_b(G, A \rtimes^u_{\alpha,N} G)$ satisfying

(a) $f(\gamma\chi) = \hat{\alpha}_\chi^{-1}\big(f(\gamma)\big)$ for $\chi \in N^\perp$, and
(b) $\gamma N^\perp \mapsto \|f(\gamma)\|$ vanishes at infinity on $\hat{G}/N^\perp$.

We would like, then, a group cohomological invariant which measures the obstruction to implementing an action $\alpha : G \to \mathrm{Aut}(A)$ with $\alpha(N) \subseteq \mathrm{Inn}(A)$ by a Green twisting map $u : N \to \mathcal{UM}(A)$. For this section, we shall not assume that G is abelian or that A is continuous-trace, but we do still require G to be second countable and A to be separable.

We proceed exactly as in the case where $\alpha(G) \subseteq \mathrm{Inn}(A)$. Since $\alpha : N \to \mathrm{Inn}(A)$ is continuous for the quotient topology on $\mathrm{Inn}(A) = \mathcal{UM}(A)/\mathcal{UZM}(A)$ [19; Corollary 0.2], we can find a Borel map $u : N \to \mathcal{UM}(A)$ such that $\alpha_n = \mathrm{Ad}\, u_n$ for all $n \in N$. As usual, there is a Borel map $\mu : N \times N \to \mathcal{UZM}(A)$ such that

$$u_m u_n = \mu(m,n) u_{mn} \qquad \text{for } m,\, n \in N, \tag{5.2}$$

and $[\mu] \in H^2\big(N, \mathcal{UM}(A)\big)$ is the obstruction $c(\alpha)$ to implementing $\alpha|_N$ by a unitary group. For $s \in G$, $n \in N$, and $a \in A$ we have

$$\begin{aligned} \mathrm{Ad}\, \alpha_s(u_{s^{-1}ns})(a) &= \alpha_s\big(\mathrm{Ad}\, u_{s^{-1}ns}\big(\alpha_s^{-1}(a)\big)\big) \\ &= \alpha_s\big(\alpha_{s^{-1}ns}\big(\alpha_{s^{-1}}(a)\big)\big) \\ &= \mathrm{Ad}\, u_n(a), \end{aligned}$$

and therefore $\alpha_s(u_{s^{-1}ns})$ and u_n differ by an element of $\mathcal{UZM}(A)$. Thus there is a Borel map $\lambda : G \times N \to \mathcal{UZM}(A)$ such that

$$\alpha_s(u_{s^{-1}ns}) = \lambda(s,n) \cdot u_n \qquad \text{for } s \in G,\, n \in N. \tag{5.3}$$

(We have chosen this rather than the more obvious comparison of $\alpha_s(u_n)$ with $u_{sns^{-1}}$ to make (5.8) below more palatable.) We can, and shall, always assume that the pair (λ,μ) has been normalized so that

$$\mu(n,e) = 1 = \mu(e,n) \tag{5.4}$$

$$\lambda(e,n) = 1 = \lambda(s,e) \tag{5.5}$$

for all $n \in N$ and $s \in G$. (This is done by insisting that $u_e = 1$.)

Lemma 5.1. *Suppose that (A,G,α) is a separable dynamical system and that N is a closed normal subgroup of G such that $\alpha(N) \subseteq \mathrm{Inn}(A)$. Choose a Borel map $u : N \to \mathcal{UM}(A)$ such that $\alpha|_N = \mathrm{Ad}\, u$ and $u_e = 1$. Define λ, μ by* (5.2) *and* (5.3)*, respectively. Then for m, n, $p \in N$ and s, $t \in G$ we have, in addition to Equations* (5.4) *and* (5.5)*,*

$$\mu(m,n)\mu(mn,p) = \mu(m,np)\mu(n,p), \tag{5.6}$$

$$\lambda(m,n) = \mu(m,m^{-1}nm)\mu(n,m)^{-1}, \tag{5.7}$$

$$\lambda(st,n) = \lambda(s,n)\alpha_s\big(\lambda(t,s^{-1}ns)\big), \text{ and} \tag{5.8}$$

$$\lambda(s,mn) = \alpha_s\big(\mu(s^{-1}ms,s^{-1}ns)\big)^{-1}\mu(m,n)\lambda(s,m)\lambda(s,n). \tag{5.9}$$

Proof. Item (5.6) is just the usual cocycle identity for μ, and to establish (5.7), expand Equation (5.3) using (5.2) and $\alpha_n = \mathrm{Ad}\, u_n$. The identities (5.8) and (5.9) follow from similar calculations. □

Remark 5.2. Condition (5.8) says that the function

$$\tilde{\lambda} : G \to C^1\big(N,\mathcal{UZM}(A)\big)$$

defined by $\tilde{\lambda}(s)(n) = \lambda(s,n)$ is a 1-cocycle for the action of G on $C^1\big(N,\mathcal{UZM}(A)\big)$ given by $(s\cdot\varphi)(n) = \alpha_s\big(\varphi(s^{-1}ns)\big)$. There is also a natural action of G on $Z^2\big(N,\mathcal{UZM}(A)\big)$ given by $s\cdot\mu(m,n) = \alpha_s\big(\mu(s^{-1}ms,s^{-1}ns)\big)$, and then Condition (5.9) says that $\partial\big(\tilde{\lambda}(s)\big) = \mu^{-1}(s\cdot\mu)$. We observe that $\tilde{\lambda}$ is a Borel cocycle if we pass to the quotient $\underline{C}^1\big(N,\mathcal{UZM}(A)\big)$ of $C^1\big(N,\mathcal{UZM}(A)\big)$. (This follows from the first part of the proof of [10; Theorem 1].)

The pairs (λ,μ) will form the cocycles in our relative cohomology group Λ. To identify the appropriate equivalence relation, we suppose that we had chosen a different Borel map $u' : N \to \mathcal{UM}(A)$ satisfying $\alpha|_N = \mathrm{Ad}\,u'$. Then there is a Borel function $\rho : N \to \mathcal{UZM}(A)$ such that $u' = \rho\cdot u$. As usual the 2-cocycles are related by $\mu' = (\partial\rho)\mu$, and the λ's by

$$\lambda'(s,n) = \alpha_s\big(\rho(s^{-1}ns)\big)\rho(n)^{-1}\lambda(s,n).$$

If we let $\Delta\rho$ denote the pair (λ_1,μ_1), where

(5.10) $$\mu_1(m,n) = \partial\rho(m,n) = \rho(m)\rho(n)\rho(mn)^{-1}, \text{ and}$$

(5.11) $$\lambda_1(s,n) = \alpha_s\big(\rho(s^{-1}ns)\big)\rho(n)^{-1},$$

then we have $(\lambda',\mu') = \Delta\rho(\lambda,\mu)$. We are now ready for the formal definition of our Λ-invariant.

Definition 5.3. Let G be a second countable locally compact group, N a closed normal subgroup, and M an abelian Polish G/N-module. For consistency, denote the action of $s \in G$ on $m \in M$ by $\alpha_s(m)$. Let $Z(G,N;M)$ denote the group of pairs (λ,μ), where $\mu : N\times N \to M$ and $\lambda : G\times N \to M$ are Borel maps satisfying Conditions (5.4)–(5.9), and the group operation is pointwise multiplication. Let $B(G,N;M)$ denote the subgroup of all pairs of the form $\Delta\rho$ (see Conditions (5.10), (5.11)) for some Borel map $\rho : N \to M$, and let $\Lambda(G,N;M)$ be the quotient $Z(G,N;M)/B(G,N;M)$.

Proposition 5.4. *Let (A,G,α) be a separable dynamical system, and suppose that N is a closed normal subgroup of G such that $\alpha(N) \subseteq \operatorname{Inn}(A)$. Define $(\lambda,\mu) \in Z\big(G,N;\mathcal{UZM}(A)\big)$ in terms of a Borel lifting u for $\alpha|_N$ by Equations (5.2) and (5.3). Then the class $d(\alpha) = [\lambda,\mu]$ of (λ,μ) in $\Lambda\big(G,N;\mathcal{UZM}(A)\big)$ is independent of any of the choices made, and vanishes if and only if there is a Green twisting map for α on N.*

Proof. We have already proved everything except the last statement. But one direction is obvious—if u is a Green twisting map, then μ and λ are identically 1. Conversely, if (λ,μ) are defined using $u : N \to \mathcal{UM}(A)$ and $(\lambda,\mu) = \Delta\rho$, then $v_n = \rho(n)^{-1}u_n$ defines a Borel homomorphism $v : N \to \mathcal{UM}(A)$, which is automatically continuous and satisfies

$$\begin{aligned}\alpha_n &= \operatorname{Ad} u_n = \operatorname{Ad} v_n, \text{ and}\\ \alpha_s(v_n) &= \alpha_s\big(\rho(n)\big)^{-1}\alpha_s(u_n) = \alpha_s\big(\rho(n)\big)^{-1}\lambda(s,sns^{-1})u_{sns^{-1}}\\ &= \rho(sns^{-1})^{-1}u_{sns^{-1}} = v_{sns^{-1}}.\end{aligned}$$ □

Remark 5.4. We have referred to the group $\Lambda(G,N;M)$ as a relative Moore cohomology group, and of course we should explain why we have done this. For *discrete* G, it has been shown by several authors that the group Λ fits into an eight term exact sequence

$$\begin{aligned}0 \longrightarrow H^1(G/N,M) \xrightarrow{\text{Inf}} H^1(G,M) \xrightarrow{\text{Res}} H^1(N,M) \longrightarrow H^2(G/N,M) \longrightarrow\\ \xrightarrow{\text{Inf}} H^2(G,M) \longrightarrow \Lambda(G,N;M) \longrightarrow H^3(G/N,M) \xrightarrow{\text{Inf}} H^3(G,M)\end{aligned}$$

(e.g., [4], [6], [8], [20]); indeed, in Loday's proof of this, he defined the relative cohomology to be that of the quotient complex $\{C^*(G,M)/C^*(G/N,M)\}$, so that there is almost by definition a long exact sequence, and then identified H^3 with Λ [8]. Huebschmann [4] and Ratcliffe [20] defined $\Lambda(G,N;M)$ to consist of the G-crossed extensions of N by M; roughly speaking, these are the ordinary group extensions with a compatible action of G. Although we shall not pursue them here, there will be similar results relating our Borel version of Λ to Moore cohomology and to Polish crossed extensions—in fact, we originally formulated our arguments in terms of Polish extensions, and one still appears in Section 8.

The relevance of Λ to group actions on operator algebras was pointed out to us by Colin Sutherland, who has been heavily involved in the classification of discrete group actions on injective von Neumann algebras (e.g., [6], [25]). If $\alpha : G \to \mathrm{Aut}(M)$ is such an action, two ingredients in the classification are the subgroup $N = \alpha^{-1}(\mathrm{Inn}(M))$ and the obstruction $d(\alpha|_N)$ to implementing α on N by a homomorphism $u : N \to \mathcal{U}(M)$ satisfying Equation (5.1). Our Borel version will not be so useful for actions of locally compact groups, since $\mathrm{Inn}(M)$ is in general not a closed subgroup of $\mathrm{Aut}(M)$; however, for continuous-trace C^*-algebras we do often have $\mathrm{Inn}(A)$ closed in $\mathrm{Aut}(A)$ [19; Theorem 0.8], and hence our invariant should be particularly relevant in this case.

6. The Λ-invariant and diamonds of bundles. We say that an element (λ,μ) of $Z\big(G,N;C(X,\mathbf{T})\big)$ is *pointwise trivial* if μ belongs to $Z^2_{PT}\big(N,C(X,\mathbf{T})\big)$, and we write $(\lambda,\mu) \in Z_{PT}\big(G,N;C(X,\mathbf{T})\big)$. Suppose that G is abelian. Our goal here is to construct from each pointwise trivial (λ,μ) a commutative diamond

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ F_{(\lambda,\mu)} \arrow[dr] & & X \arrow[dl] \\ & X/G & \end{tikzcd}$$

in which the southeast arrows are $\hat{N}$-bundles and the southwest arrows are G/N-bundles, such that $F_{(\lambda,\mu)} \to X/G$ is trivial if and only if $[\lambda,\mu] = 0$ in $\Lambda\big(G,N;C(X,\mathbf{T})\big)$. The idea is to take $E = E_\mu$, use λ to define an action of G on E which commutes with the action of $\hat{N}$, and define $F_{(\lambda,\mu)} = E/G$. We begin with two straightforward results on diamonds of bundles.

Lemma 6.1. *Suppose that H and Q are locally compact groups, that $p : X \to T$ is a Q-bundle, and that $r : E \to X$ is an H-bundle. Suppose there is a free and proper action of Q on E which commutes with the action of H and for which r is equivariant. Then the formula $h \cdot (Q \cdot \xi) = Q \cdot (h \cdot \xi)$ defines a free and proper action of H on E/Q. Further, the map $p \circ r : E \to T$ induces a continuous open surjection $q : E/Q \to T$, which in turn induces a homeomorphism of $(E/Q)/H$ onto T; in other words, $q : E/Q \to T$ is an H-bundle.*

Proof. The action of H on E/Q is well-defined because the actions on E commute, and it is free because r is equivariant, Q acts freely on X, and H acts freely on E. To see that the action is proper on E/Q, consider nets $Q \cdot \xi_k \to Q \cdot \xi$ and $h_k \cdot Q \cdot \xi_k \to Q \cdot \eta$; it will suffice to show that $\{h_k\}$ has a convergent subnet. By passing to a subnet, we may suppose that $\xi_k \to \xi$ in E, and that there are $s_k \in Q$ such that $h_k \cdot s_k \cdot \xi_k \to \eta$ in E. However, this implies that $r(\xi_k) \to r(\xi)$ and that $r(h_k \cdot s_k \cdot \xi_k) = s_k \cdot r(\xi_k) \to r(\eta)$, so the properness of the Q-action allows us to assume, by passing to yet another subnet, that $\{s_k\}$ converges to some $s \in Q$. Now we have $s_k \cdot \xi_k \to s \cdot \xi$ and $h_k \cdot (s_k \cdot \xi_k) \to \eta$, which since $\hat{N}$ acts properly on E implies that $\{h_k\}$ has a convergent subnet. Thus, H acts properly on E/Q. The map $p \circ r$ is open and constant on Q-orbits, and hence induces a continuous open map $q : E/Q \to T$ which is easily seen to be surjective. This map is constant on H-orbits since r is, and induces a bijection of $(E/Q)/H$ onto $T = (E/H)/Q$, which is a homeomorphism because q is open and continuous. □

Lemma 6.2. *Suppose we have a commutative diamond*

$$\begin{tikzcd} & E \arrow[dl, "i"'] \arrow[dr, "r"] & \\ Y \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & T & \end{tikzcd}$$

in which i and p are Q-bundles; r and q are H-bundles; and i and r are H- and Q-equivariant, respectively. Then the map $f(\xi) = \big(i(\xi), r(\xi)\big)$ is an isomorphism of the H-bundle E onto the pull back $p^*(Y)$, and carries the action of Q into that given by $q \cdot (y,x) = (y, q \cdot x)$.

Proof. This follows from Remark 2.3. □

We now return to the main construction. For the rest of this section, G will be a second countable locally compact abelian group, N a closed subgroup, and X, T second countable locally compact spaces such that $p : X \to T$ is a G/N-bundle.

Proposition 6.3. *Let $(\lambda,\mu) \in Z_{PT}\big(G,N;C(X,\mathbf{T})\big)$, and let E_μ be the $\hat{N}$-bundle of Proposition 3.1. The formula*

$$s\cdot(f,x) = \big(\lambda(s,\cdot)(s\cdot x)f, s\cdot x\big) \tag{6.1}$$

defines an action of G on E_μ, which is free and proper, and commutes with the action of $\hat{N}$ on E_μ. If we set $q\big(G\cdot(f,x)\big) = p(x)$ and $F_{(\lambda,\mu)} = E_\mu/G$, then $q : F_{(\lambda,\mu)} \to T$ is an $\hat{N}$-bundle.

Proof. We first observe that since G is abelian, Equation (5.9) implies that

$$\begin{aligned}
&\partial\big(\overline{\lambda(s,\cdot)(s\cdot x)f}\big)(m,n)\\
&\quad= \overline{\lambda(s,m)(s\cdot x)f(m)\lambda(s,n)(s\cdot x)f(n)}\lambda(s,mn)(s\cdot x)f(mn)\\
&\quad= \overline{\alpha_s\big(\mu(m,n)\big)(s\cdot x)}\mu(m,n)(s\cdot x)\partial\bar{f}(m,n)\\
&\quad= \overline{\mu(m,n)(x)}\mu(m,n)(s\cdot x)\mu(m,n)(x)\\
&\quad= b_\mu(s\cdot x)(m,n),
\end{aligned}$$

so $s\cdot(f,x)$ does belong to E_μ. Similarly, Equation (5.8) implies that $s\cdot\big(t\cdot(f,x)\big) = st\cdot(f,x)$, and Equation (6.1) does define an action of G on E_μ.

To see that this action is jointly continuous, recall that $\underline{C}^1\big(N,C(X,\mathbf{T})\big)$, topologized as in [9], is a Polish G-module for the G-action given pointwise on $C(X,\mathbf{T})$. It follows from Remark 5.2 and [9; Theorem 3] that $\tilde{\lambda}$ is continuous from G to $\underline{C}^1\big(N,C(X,\mathbf{T})\big)$. Therefore if $s_k \to s$ in G and $x_k \to x$ in X, then we need to show that $f_k \to f$ in $\underline{C}^1(N,\mathbf{T})$, where

$$f_k(n) = \tilde{\lambda}(s_k)(n)(x_k) = \lambda(s_k,n)(x_k).$$

Replacing $\{f_k\}$ by a subsequence and relabeling, it will suffice to show that $\{f_k\}$ has a subsequence converging almost everywhere on N. But by again replacing $\{f_k\}$ by a subsequence and relabeling, we can assume that $\tilde{\lambda}(s_k) \to \tilde{\lambda}(s)$ almost everywhere on N, say on $N\setminus S$ [9; Proposition 6]. Of course, then $\tilde{\lambda}(s_k)(n)(x_k) \to \tilde{\lambda}(s)(n)(x)$ for each $n \in N\setminus S$, since $C(X,\mathbf{T})$ carries the topology

of uniform convergence on compacta. It now follows that the action is jointly continuous.

Since μ is pointwise trivial, it is symmetric, and condition (5.7) of Lemma 5.1 implies that $\tilde{\lambda}(n) \equiv 1$ for all $n \in N$. Thus, our G-action has constant isotropy N. Since the action of G on "the second factor" of E_μ is already proper, we have now established that E_μ is a G/N-bundle. Because the actions of G and $\hat{N}$ commute, the result follows from Lemma 6.1. □

Proposition 6.4. *Suppose that* $(\lambda,\mu) \in Z_{PT}\big(G,N;C(X,\mathbf{T})\big)$. *Then the* $\hat{N}$*-bundle* $q : F_{(\lambda,\mu)} \to T$ *of Proposition 6.3 is trivial if and only if* $[\lambda,\mu] = 0$ *in* $\Lambda\big(G,N;C(X,\mathbf{T})\big)$.

Proof. If $(\lambda,\mu) = \Delta\rho$, then as in the proof of Proposition 3.4, $h(x) = \big(\rho(\,\cdot\,)(x)^{-1},x\big)$ is a continuous section of E_μ. Further, we have from Equation (5.11) that

$$\begin{aligned} s\cdot h(s^{-1}\cdot x) &= \big(\lambda(s,\cdot\,)(x)\rho(\,\cdot\,)(s^{-1}\cdot x), s\cdot(s^{-1}\cdot x)\big)\\ &= \big(\big[\lambda(s,\cdot\,)\rho(\,\cdot\,)^{-1}\big](x),x\big)\\ &= \big(\rho(\,\cdot\,)(x),x\big)\\ &= h(x), \end{aligned}$$

and we can therefore define a section k of $F_{(\lambda,\mu)} = E_\mu/G$ by $k(G\cdot x) = G\cdot h(x)$.

Conversely, suppose that $F_{(\lambda,\mu)} = E_\mu/G$ is trivial. It follows from Lemma 6.2 that $E_\mu \cong p^*(F_{(\lambda,\mu)})$ is trivial and has a continuous section $h : X \to E_\mu$. By Remark 3.7, h has the form

$$h(x) = \big(\rho(\,\cdot\,)(x)^{-1},x\big)$$

for some Borel map $\rho : N \to C(X,\mathbf{T})$ with $\partial\rho = \mu$. The equivariance of h implies that

$$\big(\overline{\rho(\,\cdot\,)(s\cdot x)}, s\cdot x\big) = h(s\cdot x) = s\cdot h(x) = \big(\lambda(s,\cdot\,)(s\cdot x)\overline{\rho(\,\cdot\,)(x)}, s\cdot x\big),$$

which reduces to (5.11) since G is abelian, and we have $(\lambda,\mu) = \Delta\rho$. □

Theorem 6.5. *Let G be a second countable locally compact abelian group, N a closed subgroup, and X a second countable locally compact G-space such that the orbit map $p : X \to X/G$ is a G/N-bundle. Then the map $(\lambda,\mu) \mapsto F_{(\lambda,\mu)}$ of Proposition 6.3 induces a well-defined isomorphism of $\Lambda\big(G,N;C(X,\mathbf{T})\big)$ onto*

$$\Big\{[F] \in HP(X/G,\hat{\mathcal{N}}) : (\psi_n)_*\big(p^*(F)\big) \textit{ is a trivial } \mathbf{T}\textit{-bundle for all } n \in N\Big\}.$$

Proof of all but surjectivity. It follows from the definition of $F_{(\lambda,\mu)}$ as E_μ/G and Lemma 6.2 that $p^*(F_{(\lambda,\mu)}) \cong E_\mu$, and Corollary 3.10 implies that $[F_{(\lambda,\mu)}]$ lies in the given subgroup of $HP(X/G,\hat{\mathcal{N}})$. If we can show that

$$F_{(\lambda,\mu)} * F_{(\lambda',\mu')} \cong F_{(\lambda\lambda',\mu\mu')},$$

then the argument of Proposition 3.8, using Proposition 6.4 in place of Proposition 3.4, will show that $[\lambda,\mu] \mapsto [F_{(\lambda,\mu)}]$ is a well-defined monomorphism.

Suppose, therefore, that (λ,μ) and (λ',μ') are in $Z_{PT}(G,N;C(X,\mathbf{T}))$. We start by defining a G action on the $\hat{N}$-bundle $E_\mu * E_{\mu'}$ by the formula $s\cdot[\xi,\eta] = [s\cdot\xi,s\cdot\eta]$ (this is well-defined because the G- and $\hat{N}$-actions commute). Notice that if $s\cdot[\xi,\eta] = [\xi,\eta]$, then there exists $\gamma \in \hat{N}$ such that $(s\cdot\xi,s\cdot\eta) = (\gamma\xi,\bar{\gamma}\eta)$. Because the bundle projection $r : E_\mu \to X$ is G-equivariant and $\hat{N}$-invariant, we see that such an s must be in N. Thus, $E_\mu * E_{\mu'}$ is a free G/N-space. It is also proper. To see this, suppose that $[\xi_k,\eta_k] \to [\xi,\eta]$ while $s_k\cdot[\xi_k,\eta_k] \to [\xi',\eta']$. Without loss of generality, we may assume that $(\xi_k,\eta_k) \to (\xi,\eta)$ and that there are $\gamma_k \in \hat{N}$ so that $(s_k\cdot\gamma_k\cdot\xi_k, s_k\cdot\bar{\gamma}_k\cdot\eta_k) \to (\xi',\eta')$. Then we also have $r(\xi_k) \to r(\xi)$ and $s_k\cdot r(\xi_k) = r(s_k\cdot\gamma_k\cdot\xi_k) \to r(\xi')$. It follows that $\{s_k\}$ has a convergent subsequence, and hence that $E_\mu * E_{\mu'}$ is a G/N-bundle.

Now Lemma 6.2 implies that $E_\mu * E_{\mu'}/G$ is a $\hat{N}$-bundle over X/G. Define $\psi : E_\mu * E_{\mu'} \to F_{(\lambda,\mu)} * F_{(\lambda',\mu')}$ by

$$\psi\big((f,x),(g,x)\big) = \big(G\cdot(f,x),G\cdot(g,x)\big).$$

Since ψ is continuous, constant on G-orbits, $\hat{N}$-equivariant, and induces the identity on X/G, ψ implements an isomorphism of $E_\mu * E_{\mu'}/G$ onto $F_{(\lambda,\mu)} * F_{(\lambda',\mu')}$. On the other hand, we have already observed that the map $\big((f,x),(g,x)\big) \mapsto (fg,x)$ induces an isomorphism φ of $E_\mu * E_{\mu'}$ onto $E_{\mu\mu'}$ (see the proof of Proposition 3.8), and since φ is G-equivariant, it implements an isomorphism of $E_\mu * E_{\mu'}/G$ and $F_{(\lambda\lambda',\mu\mu')}$. This completes the proof that Ψ is an injective homomorphism. □

7. Actions which are proper modulo N. Let (A,G,α) be a separable dynamical system in which A is a continuous-trace algebra with spectrum X, G is abelian, α is pointwise unitary on a closed subgroup N of G, and $p : X \to X/G$ is a G/N-bundle; we shall sum this up by saying that α is *proper modulo N*. We now want to show that if in addition $\alpha(N) \subseteq \mathrm{Inn}(A)$, then the construction of the previous section connects up the algebraic and topological invariants associated with α. From this we shall deduce our structure theorems for actions which are proper modulo N.

Proposition 7.1. *Suppose that (A,G,α) is a dynamical system such that α is proper modulo N, $\alpha(N) \subseteq \mathrm{Inn}(A)$, and $\hat{A} = X$. If $q(\alpha)$ is the class of $q : (A \rtimes_\alpha G)^\wedge \to X/G$ in $HP(X/G,\hat{\mathbf{N}})$ ([13; Corollary 2.1]), $d(\alpha)$ is the class in $\Lambda_{PT}(G,N;C(X,\mathbf{T}))$ corresponding to (A,G,α), and Ψ is the isomorphism defined in Theorem 6.5, then $\Psi(d(\alpha)) = q(\alpha)$.*

Proof. Recall that we obtain a representative (λ,μ) for $d(\alpha)$ by choosing a Borel map $u : N \to \mathcal{UM}(A)$ satisfying $\alpha_n(a) = u_n a u_n^*$ for all $n \in N$ and $a \in A$, and using Equations (5.2) and (5.3) to determine (λ,μ). We resume the notation of Proposition 3.8. As in that Proposition, each $(f,x) \in E_\mu$ determines a unitary representation $fu(x)$ of N which is defined by $n \mapsto f(n)u_n(x)$. Furthermore, $(\varepsilon_x, fu(x))$ is an irreducible covariant representation of $A \rtimes_\alpha N$, and there is an $\hat{N}$-bundle isomorphism $h : E_\mu \to (A \rtimes_\alpha N)^\wedge$ defined by $(f,x) \mapsto (\varepsilon_x, fu(x))$. For the moment, fix $s \in G$ and $x \in X$. We have

$$h(s \cdot (f,x)) = (\varepsilon_{s\cdot x}, f\alpha_s(u)(s \cdot x)),$$

where $f\alpha_s(u)(s \cdot x)$ stands for the representation which coincides almost everywhere with $n \mapsto f(n)\alpha_s(u_n)(s \cdot x)$. Recall that $s \cdot \varepsilon_x$ is defined by $a \mapsto \varepsilon_x(\alpha_s^{-1}(a))$. By assumption, $s \cdot \varepsilon_x$ is unitarily equivalent to $\varepsilon_{s\cdot x}$: let V be a unitary which implements the equivalence, so that

$$Va(s \cdot x)V^* = \alpha_s^{-1}(a)(x)$$

for $a \in A$. Now one computes that V implements an equivalence between the representations $L = (\varepsilon_{s\cdot x}, f\alpha_s(u)(s \cdot x))$ and $M = (s \cdot \varepsilon_x, fu(x))$. That is,

$$h(s \cdot (f,x)) = [s \cdot \varepsilon_x, fu(x)].$$

It follows from the proof of [19; Proposition 2.2] that h is G-equivariant from E_μ to $(A \rtimes_\alpha N)^\wedge$ and induces the identity on X/G. Thus the proposition follows from Remark 2.3 and [19; Proposition 2.2]. □

Theorem 7.2. *Suppose that (A,G,α) is a dynamical system such that α is proper modulo N and $\hat{A} = X$. Then $(A \rtimes_\alpha G)^\wedge$ is a trivial $\hat{N}$-bundle over X/G if and only if there is a Green twisting map for α over N.*

Proof. By [13; Proposition 2.1] and Lemma 6.2, $(A \rtimes_\alpha N)^\wedge$ is isomorphic to the pull-back of $(A \rtimes_\alpha G)^\wedge$ over the orbit map $p : X \to X/G$. Therefore, if $(A \rtimes_\alpha G)^\wedge$ is a trivial bundle, then so is $(A \rtimes_\alpha N)^\wedge$. It then follows from [13; Corollary 1.11] that $\alpha(N) \subseteq \mathrm{Inn}(A)$. Thus Proposition 7.1 applies and the result follows from Proposition 6.4 and Proposition 5.4. On the other hand, if there is a Green twisting map for α over N, then by definition $\alpha(N) \subseteq \mathrm{Inn}(A)$. Therefore the converse also follows from Propositions 7.1, 6.4, and 5.4. □

Corollary 7.3. *Let (A,G,α) be a dynamical system such that α is proper modulo N and $\hat{A} = X$. Then $(A \rtimes_\alpha G)^\wedge$ is trivial as an $\hat{N}$-bundle over X/G if and only if there is a continuous-trace C^*-algebra B with spectrum X/G and a pointwise unitary action β of $N^\perp$ on B such that $(A \rtimes_\alpha G,\hat{G},\hat{\alpha})$ is covariantly isomorphic to $\big(\mathrm{Ind}_{N^\perp}^{\hat{G}}(B,\beta),\hat{G},\tau\big)$.*

Proof. Since $\mathrm{Ind}_{N^\perp}^{\hat{G}}(B,\beta)$ has spectrum $(\hat{G}/N^\perp) \times \hat{B} = \hat{N} \times (X/G)$ (e.g., [19; Proposition 3.2]), one direction is easy. Conversely, if $(A \rtimes_\alpha G)^\wedge$ is trivial, the theorem implies that there is a Green twisting map $u : N \to \mathcal{UM}(A)$ for α. By [12; Theorem 2.4], there is then an isomorphism

$$A \rtimes_\alpha G \cong \mathrm{Ind}_{N^\perp}^{\hat{G}}(A \rtimes_{\alpha,N}^u G,\hat{\alpha}|_{N^\perp})$$

which carries the dual action of $\hat{G}$ into the canonical action τ of $\hat{G}$ by translation on the induced algebra. Thus it only remains to verify that $A \rtimes_{\alpha,N}^u G$ is a continuous-trace algebra with spectrum X/G, and that $\hat{\alpha}$ is pointwise unitary on $N^\perp$.

The twisted covariant system (A,G,N,α,u) is essentially free in the sense of Green [1], and hence it follows from Theorem 24 of [1] that $A \rtimes_{\alpha,N}^u G$ has spectrum X/G; it has continuous trace because it is a quotient of $A \rtimes_\alpha G$, which has continuous trace by [19; Corollary 2.5 (2)]. It also follows from [1; Theorem 24] that every irreducible representation of $A \rtimes_{\alpha,N}^u G$ is equivalent to one of the form $\mathrm{Ind}_N^G(\pi)$, where π is an irreducible representation of $A \cong A \rtimes_{\alpha,N}^u N$. The covariant representation of (A,G,α) corresponding to $\mathrm{Ind}(\pi)$ can be identified with $(\tilde{\pi},\lambda)$, acting on the Hilbert space $\mathcal{H}$ consisting of those Borel functions $f : G \to \mathcal{H}_\pi$ which satisfy

$$f(sn) = \pi(n)^{-1}\big(f(s)\big) \text{ for } s \in G,\, n \in N \quad \text{and} \quad \int_{G/N} \|f(s)\|^2 \, d(sN) < \infty,$$

according to the formulas

$$\tilde{\pi}(a)f(t) = \pi\big(\alpha_t^{-1}(a)\big)\big(f(t)\big), \quad \text{and} \quad \lambda_s(f)(t) = f(s^{-1}t).$$

(It is easy to check directly that this covariant representation preserves the twist—i.e., that $\tilde{\pi} \circ u = \lambda|_N$—and hence gives a representation of $A \rtimes_{\alpha,N}^u G$.) The dual action of $\hat{\alpha}_\gamma$ fixes the copy $i_A(A)$ of A in $\mathcal{M}(A \rtimes_\alpha G)$, and multiplies the generators $i_G(s)$ for $s \in G$ by $\gamma(s)$. Thus we may define $U : N^\perp \to \mathcal{H}$ by $U_\gamma(f)(s) = \gamma(s)f(s)$, and verify easily that for $\gamma \in \hat{N}$, $a \in A$, and $s \in G$, we have

$$\tilde{\pi} \times \lambda\big(\hat{\alpha}_\gamma\big(i_A(a)\big)\big) = \tilde{\pi}\big(i_A(a)\big) = U_\gamma \tilde{\pi}\big(i_A(a)\big)U_\gamma^* = U_\gamma \tilde{\pi} \times \lambda\big(i_A(a)\big)U_\gamma^*, \text{ and}$$

$$\tilde{\pi} \times \lambda\big(\hat{\alpha}_\gamma\big(i_G(s)\big)\big) = \gamma(s)\lambda_s = U_\gamma \lambda_s U_\gamma^* = U_\gamma \tilde{\pi} \times \lambda\big(i_G(s)\big)U_\gamma^*.$$

The action $\hat{\alpha}|_{N^\perp}$ is therefore pointwise unitary, and the Corollary is proved. □

Corollary 7.4. *Let (A,G,α) be a dynamical system such that α is proper modulo N and $\hat{A} = X$. Suppose that $p : X \to X/G$ is trivial as a G/N-bundle. Then there is a pointwise unitary action ϑ of N on a continuous-trace algebra B with spectrum X/G such that $\big(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \operatorname{Ad}\rho\big)$ is covariantly isomorphic to $\big(\operatorname{Ind}_N^G(B,\vartheta), G, \tau\big)$. (So that essentially the only examples with $\hat{A}$ trivial are the ones studied in [19; Section 3(a)].)*

Proof. We apply the previous corollary to the dual system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$: we know from [13; Proposition 2.1] that $\hat{G}/N^\perp = \hat{N}$ acts freely and properly on $(A \rtimes_\alpha G)^\wedge$, so the only point to check is that $\hat{\alpha}$ is pointwise unitary on $N^\perp$. But [19; Proposition 2.1] implies in particular that every irreducible representation of $A \rtimes_\alpha G$ has the form $\operatorname{Ind}_N^G(\pi \times U)$ for some $\pi \times U \in (A \rtimes_\alpha N)^\wedge$, and the argument in the proof of the previous corollary carries over verbatim. We can therefore deduce that

$$\big((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}, G, \hat{\hat{\alpha}}\big) \cong \big(\operatorname{Ind}_N^G(B,\vartheta), G, \tau\big),$$

and the result follows from the Takai duality theorem [17; Theorem 7.9.3]. □

8. Surjectivity. In this section we merely want to fill in the remaining bit of the proof of Theorem 6.5—namely, we need to show that the map Ψ defined in Theorem 6.5 is surjective. Precisely, we must show that, given a G/N-bundle $p : X \to X/G$ and a $\hat{N}$-bundle $q : F \to X/G$, such that the pull-back $E = p^*(F)$ has the property that $(\psi_n)_*(E)$ is a trivial $\mathbf{T}$-bundle for each $n \in N$, then there are invariants (λ,μ) representing a class in $\Lambda_{PT}\big(G,N;C(X,\mathbf{T})\big)$ with $F_{(\lambda,\mu)}$ isomorphic to F as $\hat{N}$-bundles. Equivalently, we will show that E is isomorphic to E_μ both as a $\hat{N}$- and as a G/N-bundle.

First observe that $\mathbf{T} \times N \times E$ becomes a $\hat{N}$-bundle when given the action

$$\gamma \cdot (t,n,\xi) = \big(\overline{\gamma(n)}t, n, \gamma \cdot \xi\big).$$

The quotient B is a $\mathbf{T}$-bundle over $N \times X$ with the bundle map given by $h\big([t,n,\xi]\big) = \big(n, r(\xi)\big)$. Define continuous maps $\kappa : B \to X$ and $\delta : B \to N$ by the formulas

$$\kappa\big([t,n,\xi]\big) = r(\xi), \quad \text{and} \quad \delta\big([t,n,\xi]\big) = n.$$

Finally, notice that the spaces

$$B_n = h^{-1}\big(\{(n,x) : x \in X\}\big)$$

are easily seen to be isomorphic to $(\psi_n)_*(E)$. Since the latter space is a trivial $\mathbf{T}$-bundle over X, there will be continuous sections. The next lemma will be useful for describing these sections. We shall need the extra generality later.

Lemma 8.1. *Let $B' = E \underset{X}{\times} B = \{(\xi,b) \in E \times B : r(\xi) = \kappa(b)\}$. For each $(\xi,b) \in B'$, define $\vartheta(\xi,b)$ to be the unique value in $\mathbf{T}$ such that*

$$\big[\vartheta(\xi,b),\delta(b),\xi\big] = b.$$

Then $\vartheta : B' \to \mathbf{T}$ is continuous.

Proof. Suppose that $\{(\xi_k,b_k)\}$ converges to (ξ,b): it suffices to show that every subsequence of $\{\vartheta(\xi_k,b_k)\}$ has a convergent subsequence. Thus we may as well assume that there are $\gamma_k \in \hat{N}$ such that $\gamma_k \cdot \big(\vartheta(\xi_k,\xi_k),\delta(b_k),\xi_k\big) \to \big(\vartheta(\xi,b),\delta(b),\xi\big)$. Now as $\xi_k \to \xi$ and $\gamma_k \cdot \xi_k \to \xi$, we may also assume that $\gamma_k \to 1$. Since $\delta(b_k) \to \delta(b)$, it follows from

$$\overline{\gamma_k\big(\delta(b_k)\big)}\vartheta(\xi_k,b_k) \to \vartheta(\xi,b)$$

that $\vartheta(\xi_k,b_k) \to \vartheta(\xi,b)$ as well. □

Definition 8.2. Let C denote the collection of continuous functions $c : X \to B$ which satisfy

(1) $\kappa\big(c(x)\big) = x$, for all $x \in X$; and
(2) there exists $n_0 \in N$ so that $\delta\big(c(x)\big) = n_0$ for all $x \in X$.

We give C the relative topology as a subspace of $C(X,B)$ with the compact-open topology.

Lemma 8.3. *Suppose that $f : E \to \mathbf{T}$ is continuous and there exists $n_f \in N$ such that*

$$f(\gamma \cdot \xi) = \overline{\gamma(n_f)}f(\xi) \qquad \text{for all } \gamma \in \hat{N} \text{ and } \xi \in E. \tag{8.1}$$

(1) *The function $c_f : X \to B$ defined by*

$$c_f\big(r(\xi)\big) = \big[\big(f(\xi),n_f,\xi\big)\big]$$

belongs to C, and every element of C has this form for some continuous $f : E \to \mathbf{T}$ satisfying Equation (8.1).
(2) *A sequence $\{c_{f_k}\}$ converges to c_f in C if and only if $n_{f_k} \to n_f$ and $f_k \to f$ uniformly on compacta in E.*
(3) *Given $n \in N$, there is a $c_f \in C$ with $n_f = \delta(c_f) = n$.*

Proof. It is easy to verify that $c_f \in C$. To see that every $c \in C$ has this form we just apply Lemma 8.1 to the subset $\{(\xi, c(r(\xi)))\}$ of B', and define f by $f(\xi) = \vartheta(\xi, c(r(\xi)))$. Part (3) follows immediately from part (1) and the observation that each B_n is a trivial $\mathbf{T}$-bundle. Now suppose $c_{f_k} \to c_f$ in C, and $\xi_k \to \xi$ is an arbitrary convergent sequence in E. Then

$$f_k(\xi_k) = \vartheta(\xi_k, c_{f_k}(r(\xi_k))) \to \vartheta(\xi, c_f(r(\xi))) = f(\xi),$$

and hence $f_k \to f$ uniformly on compacta. Since we trivially have that the constant value n_{f_k} of $\delta \circ c_{f_k}$ converges to n_f, this gives one implication of part (2). Since any compact set in X has a compact pre-image in E, the converse implication is straightforward. □

In view of this lemma we can define a group structure on C by the rules $c_f c_g = c_{fg}$ and $c_f^{-1} = c_{\bar{f}}$, where

$$c_{fg}(r(\xi)) = [f(\xi)g(\xi), n_f n_g, \xi], \text{ and}$$
$$c_{\bar{f}}(r(\xi)) = [\overline{f(\xi)}, n_f^{-1}, \xi],$$

and these operations are continuous for the topology of $C(X,B)$. Since C is clearly a closed subspace of the Polish space $C(X,B)$, we have shown that C is an abelian Polish group.

Now it is evident from Lemma 8.3 that $\delta|_C$ is a continuous surjection of C onto N with kernel $i(C(X,\mathbf{T}))$, where $i : C(X,\mathbf{T}) \to C$ is the continuous injection defined by

$$i(\varphi)(r(\xi)) = [\varphi(r(\xi)), 1, \xi].$$

It is a consequence of [9; Proposition 5] that

$$1 \longrightarrow C(X,\mathbf{T}) \xrightarrow{\ i\ } C \xrightarrow{\ \delta\ } N \longrightarrow 1 \tag{8.1}$$

is an exact sequence of Polish groups.

Since C is Polish, we can choose a Borel cross section σ for δ. Define $g : N \times E \to \mathbf{T}$ by

$$\sigma(n)(r(\xi)) = [g(n,\xi), n, \xi].$$

In the notation of Lemma 8.1, $g(n,\xi) = \vartheta(\xi, \sigma(n)(r(\xi)))$. Therefore, we can conclude that g is Borel, that g is continuous in the second variable, and that g satisfies

$$g(n, \gamma \cdot \xi) = \overline{\gamma(n)} g(n,\xi)$$

for each $n \in N$, $\xi \in E$, and $\gamma \in \hat{N}$. Now we can define $\mu : N \times N \to C(X,\mathbf{T})$ and $\lambda : G \times N \to C(X,\mathbf{T})$ by

$$\mu(n,m)\big(r(\xi)\big) = g(n,\xi)g(m,\xi)\overline{g(nm,\xi)} \tag{8.3}$$

$$\lambda(s,n)\big(r(\xi)\big) = \overline{g(n,\xi)}g(n,s^{-1}\cdot\xi). \tag{8.4}$$

It is a routine matter to verify that (λ,μ) satisfy conditions (5.6)–(5.9). Furthermore, μ is symmetric, and hence is pointwise trivial [7], so that (λ,μ) represents a class in $\Lambda_{PT}\big(G,N;C(X,\mathbf{T})\big)$. Finally, we can define $\Theta : E \to E_\mu$ by

$$\Theta(e) = \big(\overline{g(\,\cdot\,,e)},r(e)\big).$$

Since the action of G on E_μ is defined in terms of the λ given by Equation (8.4), one can verify that Θ is G-invariant as well as $\hat{N}$-invariant; since

$$\begin{tikzcd} E \arrow[rr, "\Theta"] \arrow[dr, "r"'] & & E_\mu \arrow[dl, "r_\mu"] \\ & X & \end{tikzcd}$$

commutes, Θ will be the required isomorphism provided it is continuous (Remark 2.3). However, since g is continuous in its second variable, $\xi_k \to \xi$ implies that $g(\,\cdot\,,\xi_k) \to g(\,\cdot\,,\xi)$ pointwise, and hence in $\underline{C}^1(N,\mathbf{T})$. This completes the proof that Ψ is surjective. □

Remark 8.4. The Polish group C carries a natural action of G, given by $s\cdot c(x) = s\cdot\big(c(s^{-1}\cdot x)\big)$, and the extension (8.2) is then a G-crossed extension of N by $C(X,\mathbf{T})$. Thus although we have deliberately chosen to work in terms of cocycles, a Polish version of the theory of crossed extensions is lurking close by.

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