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Abstract Some recent efforts to reformulate analytic number theory in terms of Hamiltonian eigenspectra has led to some developments in non-Hermitian operator theory. Herein we examine analytic number theory using Hamilton's equations from quantum mechanics. Borrowing from the second axiom of Kolmogorov, the eigenfunctions of these equations can be treated as a chaotic quantum system in a rigged Hilbert space, much like the harmonic oscillator is for integrable quantum systems. As such, herein we perform a symmetrization procedure from the recent developments of non-Hermitian operator theory to obtain a Hermitian analogue using a similarity transformation, and provide an analytical expression for the eigenvalues of the results using Green's functions. A nontrivial expression for the eigensolution of the Hamilton eigenequation is also obtained. A Gelfand triplet is then used to ensure that the eigensolution is well defined. The holomorphicity of the resulting eigenspectrum is demonstrated, and a second quantization of the resulting Schrödinger equation is performed. From the holomorphicity of the eigensolution, a general solution is also obtained by performing an invariant similarity transformation.

1 Introduction

The unification of number theory with quantum mechanics has been the subject of many research investigations [1–5]. It has been proven that an infinitude of prime numbers exist [7]. In Refs. [8,9], it was shown that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator may have implications for analytic number theory. Although the well-posedness of the BBM conjecture was disputed by Jean Bellissard [10], his concerns were refuted in Ref. [11]. According to the Hilbert-Pólya conjecture, if the BBM conjecture

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is correct [12], the eigenvalues can be considered as the spectrum of an operator $\hat{R} = \hat{I}/2 + i\hat{H}$, where $\hat{H}$ is a self-adjoint Hamiltonian operator [5,6, 13], and $\hat{I}$ is identity. This is similar to what Hilbert proposed as the eighth problem on a list of significant mathematics problems [14]. Although the BBM Hamiltonian is pseudo-Hermitian [15], it is consistent with the Berry-Keating conjecture [16–18], which states that when $\hat{x}$ and $\hat{p}$ commute, the Hamiltonian reduces to the classical $H = 2xp$. Similarly, Berry, Keating, and Connes proposed a classical Hamiltonian in order to study the problem. Recently, the classical Berry-Keating Hamiltonians were quantized, and were shown to smoothly approximate the eigenspectra [19,20]. This reformulation was found to be physically equivalent to the Dirac equation in Rindler spacetime [21]. Herein, the eigenvalues of the BBM Hamiltonian are taken to be the imaginary parts of the analytical continuation of the eigenfunction

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{1}$$

where the complex number $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2 + t^2}$, and $\theta = \arctan(t/\sigma)$, $\Re(s) > 0$. The idea that the imaginary parts of Eq. (1) can be obtained by a self-adjoint operator was conjectured by Hilbert and Pólya [22]. Hilbert and Pólya asserted that Eq. (1) can be studied with a self-adjoint operator in a suitable Hilbert space. The Hilbert-Pólya conjecture has also found applications in quantum field theories [23]. In Ref. [25], Hardy proved that infinitely many zeros are located at $\sigma = 1/2$ [12,24]. According to the Prime Number Theorem [26,27], no zeros of Eq. (1) can exist at $\sigma = 1$. The paper is organized as follows: In Sec. 2 we present a Schrödinger equation whose eigenfunctions are identical to those of the BBM Hamiltonian and evaluate the convergence of the solution by studying the orthonormality using the second axiom of Kolmogorov. A self-adjoint Hamiltonian is derived from the BBM Hamiltonian using a similarity transformation [28,29], and a second quantization of the resulting Schrödinger equation is then performed to obtain the equations of motion. Moreover, we study the holomorphic eigenvalues of the eigenfunction by taking the expectation values of the resulting Schrödinger operator. Finally we obtain a general solution to the analytic Schrödinger equation by performing a similarity transformation in Sec. 3, and make concluding remarks in Sec. 4.

1.1 Preliminaries

Definition 11 *The complex valued function (eigenstate) $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) : \mathbb{X} \to \mathbb{C}$ is measurable if $\mathbb{E}$ is a measurable subset of the measure space $\mathbb{X}$ and for each real number r, the sets $\{x \in \mathbb{E} : \phi_\sigma(x) > r\}$ and $\{x \in \mathbb{E} : \phi_t(x) > r\}$ are measurable for $\sigma, t \in \mathbb{R}$ [30].*

Definition 12 *Let ϕ_s be a complex-valued eigenstate on a measure space $\mathbb{X}$, and $\phi_s = \phi_\sigma + i\phi_t$, with ϕ_σ and ϕ_t real. Therefore, ϕ_s is measurable iff ϕ_σ and ϕ_t are measurable (Ibid.).*

Definition 13 *Suppose μ is a measure on the measure space $\mathbb{X}$, and $\mathbb{E}$ is a measurable subset of the measure space $\mathbb{X}$, and ϕ_s is a complex-valued eigenstate on $\mathbb{X}$. It follows that $\phi_s \in (\mathscr{H} = \mathscr{L}(\mu))$ on $\mathbb{E}$, and ϕ_s is complex square-integrable, if ϕ_s is measurable and (Ibid.)*

$$\int_{\mathbb{E}} \mid \phi_s \mid d\mu < +\infty. \tag{2}$$

Definition 14 *The complex valued function (eigenstate) $\phi_s = \phi_\sigma + i\phi_t$ defined on the measurable subset $\mathbb{E}$ is said to be integrable if ϕ_σ and ϕ_t are integrable for $\sigma, t \in \mathbb{R}$, where μ is a measure on the measure space $\mathbb{X}$. The Lebesgue integral of ϕ_s is defined by (Ibid.)*

$$\int_{\mathbb{E}} \phi_s d\mu = \int_{\mathbb{E}} \phi_\sigma d\mu + i \int_{\mathbb{E}} \phi_t d\mu. \tag{3}$$

Definition 15 *Let $\mathbb{X}$ be a measure space, and $\mathbb{E}$ be a measurable subset of $\mathbb{X}$. Given the complex eigenstate ϕ_s, then $\phi_s \in (\mathscr{H} = \mathscr{L}^2(\mu))$ on $\mathbb{E}$ if ϕ_s is Lebesgue measurable and if*

$$\int_{\mathbb{E}} \mid \phi_s \mid^2 d\mu < +\infty, \tag{4}$$

such that ϕ_s is square-integrable. For $\phi_s \in (\mathscr{H} = \mathscr{L}^2(\mu))$ we define the $\mathscr{L}^2$-norm of ϕ_s as

$$\parallel \phi_s \parallel_2 = \left(\int_{\mathbb{E}} \mid \phi_s \mid^2 d\mu \right)^{1/2}, \tag{5}$$

where μ is the measure on the measure space $\mathbb{X}$ (Ibid.).

Definition 16 *Let $\mathbb{X}$ be a measure space, and $\mathbb{E}$ be a measurable subset of $\mathbb{X}$. Given the complex eigenstate ϕ_s, then $\phi_s \in (\mathscr{H} = \mathscr{L}^p(\mu))$ on $\mathbb{E}$ if ϕ_s is Lebesgue measurable and if*

$$\int_{\mathbb{E}} \mid \phi_s \mid^p d\mu < +\infty, \tag{6}$$

such that ϕ_s is p-integrable. For $\phi_s \in (\mathscr{H} = \mathscr{L}^p(\mu))$ we define the $\mathscr{L}^p$-norm of ϕ_s as

$$\parallel \phi_s \parallel_p = \left(\int_{\mathbb{E}} \mid \phi_s \mid^p d\mu \right)^{1/p}, \tag{7}$$

where μ is the measure on the measure space $\mathbb{X}$ (Ibid.).

Definition 17 *A rigged Hilbert space (i.e., a Gelfand triplet [31]) is a triplet $(\Phi, \mathscr{H}, \Phi^*)$, where Φ is a dense subspace of $\mathscr{H}$ and Φ^* is its continuous dual space.*

Definition 18 *In the theory of computation, an observable is called computable, or effective, if and only if its behavior is given by a computable function [32].*

Definition 19 *Observables, e.g. $\hat{x}$ and $\hat{p}$ of a system, are represented in quantum mechanics by self-adjoint operators (which we will not notationally distinguish from the observables themselves). If there exists an observable C such that $C = \alpha\hat{x} + \beta\hat{p}$, and if $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ denote the expectation values of $\hat{x}$ and $\hat{p}$ respectively, then $\langle C\rangle = \alpha\langle\hat{x}\rangle + \beta\langle\hat{p}\rangle$ is the expectation value of C. According to Heisenberg's uncertainty principle, if the observables corresponding to two quantities $\hat{x}$ and $\hat{p}$ do not commute, i.e. $[\hat{x},\hat{p}] \neq 0$, both quantities cannot simultaneously be measured to arbitrary accuracy [33].*

Definition 110 *A linear operator $\hat{H}$ is Hermitian (self-adjoint) if it is defined on a linear everywhere-dense set $\mathscr{D}(\hat{H})$ in a Hilbert space $\mathscr{H}$ coinciding with its adjoint operator $\hat{H}^\dagger$, that is, such that $\mathscr{D}(\hat{H}) = \mathscr{D}(\hat{H}^\dagger)$ and*

$$\langle\hat{H}x, y\rangle = \langle x, \hat{H}y\rangle \tag{8}$$

for every $x, y \in \mathscr{D}(\hat{H})$ [34–36].

2 The Measurement Problem

We consider the eigenvalues of the Hamiltonian

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}), \tag{9}$$

where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$. For the Hamiltonian operator as given by Eq. (9), the Hilbert space is $\mathscr{H} = \mathscr{L}^{p=2}[1,\infty)$. In Refs. [8,9], it is conjectured that if the BBM conjecture is correct, the eigenvalues of Eq. (9) are non-degenerate. Next, we let $\Psi_s(x)$ be an eigenstate of Eq. (9) with an eigenvalue $t = i(2s-1)$, such that

$$\hat{H}\,|\Psi_s(x)\rangle = t\,|\Psi_s(x)\rangle\,, \tag{10}$$

and $x \in \mathbb{R}^+$, $s \in \mathbb{C}$. The system is described by a Hilbert space

$$\mathscr{H} = \bigotimes_{j=1}^{n} \mathscr{H}_j, \tag{11}$$

from the tensor product of infinite dimensional Fock spaces $\mathscr{H}_j$. These Fock spaces are annihilated, and created, respectively by $\hat{a}_j$ and $\hat{a}_j^\dagger$, where

$$\hat{a}_j = \frac{1}{\sqrt{2}}(x_j + \hbar\partial_{x_j}) \tag{12a}$$

$$\hat{x}_j = (\hat{a}_j + \hat{a}_j^\dagger) \tag{12b}$$

$$\hat{p}_j = (\hat{a}_j - \hat{a}_j^\dagger)/i \tag{12c}$$

for the canonical coordinates $\hat{x}_j$, $\hat{p}_j$. As such, the Bose commutation relations are satisfied

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}. \tag{13}$$

Letting $\hat{\Phi} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$ denote the vector of canonical coordinates, we then obtain the canonical commutation relations in symplectic form

$$[\hat{\Phi}_j, \hat{\Phi}_k] = 2i\omega_{jk} = 2i \bigotimes_{j=1}^{n} \boldsymbol{\omega}, \tag{14}$$

where ω_{jk} is an antisymmetric matrix, i.e., $\boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathrm{T}}$ [37]. For non-normalized eigenvectors $|\Psi_s(x)\rangle$ of the quadrature operators $\{\hat{x}_j\}$

$$\hat{x}_j \, |\Psi_s(x)\rangle = x_j \, |\Psi_s(x)\rangle \, , \tag{15}$$

where $x \in \mathbb{R}^n$ for $(j = 1, \ldots, n)$, i.e. $|\Psi_s(x)\rangle$ is an eigenstate of the operator $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ and $\hat{x}_j$ is multiplication by x_j. Similarly, for non-normalized eigenvectors $|\Psi_s(x)\rangle$ of the quadrature operators $\{\hat{p}_j\}$

$$\hat{p}_j \, |\Psi_s(x)\rangle = -i\hbar\partial_{x_j} \, |\Psi_s(x)\rangle \, , \tag{16}$$

where $|\Psi_s(x)\rangle$ is an eigenstate of the operator $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ and $\hat{p}_j$ is the operation $-i\hbar\partial_{x_j}$. Solutions to Eq. (10) are given by the analytic continuation of the Hurwitz zeta function

$$\begin{aligned} |\Psi_s(x)\rangle &= -\zeta(s, x+1) \\ &= -\Gamma(1-s)\frac{1}{2\pi i}\oint_C \frac{z^{s-1}e^{(x+1)z}}{1-e^z}dz, \end{aligned} \tag{17}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2s-1)$, $s \in \mathbb{C}$, $\Re(s) \leq 1$, the contour C is a loop around the negative real axis, and Γ is the Euler gamma function for $\Re(s) > 0$

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx. \tag{18}$$

As $-\,|\Psi_s(x=1)\rangle$ is $1 - \zeta(s^*)$, this implies that s belongs to the discrete set of eigenspectra of the eigenfunction when $s^* = \sigma - it = |s|\exp(-i\theta)$. As $-\,|\Psi_s(x=-1)\rangle$ is $\zeta(s)$, this implies that s belongs to the discrete set of eigenspectra of the eigenfunction when $s = \sigma + it = |s|\exp(i\theta)$ and $\sigma = 1/2$. We are interested in the case when $x \geq 1$, so we will focus on the positive-valued orthonormalization $x = 1$. From inserting Eq. (10) into Eq. (9), we have the relation

$$\frac{1}{1-e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) \, |\Psi_s(x)\rangle = t \, |\Psi_s(x)\rangle \, . \tag{19}$$

Given that Eq. (9) is not Hermitian, it is useful to symmetrize the system. This can be accomplished by letting

$$\begin{aligned} |\phi_s(x)\rangle &= [1 - \exp(-\partial_x)] \, |\Psi_s(x)\rangle \, , \\ &= \hat{\Delta} \, |\Psi_s(x)\rangle \\ &= |\Psi_s(x)\rangle - |\Psi_s(x-1)\rangle \, , \end{aligned} \tag{20}$$

and defining a shift operator

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x). \tag{21}$$

For $s > 0$ the only singularity of $\zeta(s,x)$ in the range of $0 \leq x \leq 1$ is located at $x = 0$, behaving as x^{-s}. More specifically,

$$\zeta(s, x+1) = \zeta(s,x) - \frac{1}{x^s}, \tag{22}$$

with $\zeta(s,x)$ finite for $x \geq 1$ [38]. As such, it can be seen from Eq. (20) that the Berry-Keating eigenfunction [16,17]

$$\begin{aligned} |\phi_s(x)\rangle &= \frac{1}{x^s} \\ &= \exp\Big(\ln(x)(-\sigma - it)\Big) \\ &= \exp\Big(-\sigma\ln(x) - it\ln(x))\Big) \\ &= \exp\Big(-\sigma\ln(x)\Big)\Big(\cos(t\ln(x)) - i\sin(t\ln(x))\Big) \\ &= x^{-\sigma}\Big(\cos(t\ln(x)) - i\sin(t\ln(x))\Big). \end{aligned} \tag{23}$$

Furthermore, the distributional orthonormality relation at $x = 1$ is satisfied such that [40]

$$\langle \phi_s | \phi_{s'} \rangle = \delta_{ss'}. \tag{24}$$

Upon inserting Eq. (20) into Eq. (19) we obtain

$$-i[x\partial_x + \partial_x x]\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{25}$$

Let $\mathscr{H}$ be a Hilbert space, and from Eq. (25) we have the Hamiltonian operator

$$\begin{aligned} \hat{H} &= -i\hbar\Big[x\partial_x + \partial_x x\Big] \\ &= -i\hbar\Big[2x\partial_x + 1\Big], \end{aligned} \tag{26}$$

for $x \in \mathbb{R}$ acting in $\mathscr{H}$, such that

$$\langle \hat{H}x, y\rangle = \langle x, \hat{H}y\rangle \;\; \forall\, x, y \in \mathscr{D}(\hat{H}). \tag{27}$$

For the Hamiltonian operator as given by Eq. (26), the Hilbert space is $\mathscr{H} = \mathscr{L}^{p=2}[1,\infty)$ [41,42,40]. Restricting $x \in \mathbb{R}^+$, Eq. (26) is then written

$$\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}, \tag{28}$$

where $s \in \mathbb{C}$, and $x \in \mathbb{R}^+$. For the Hamiltonian operator as given by Eq. (28), the Hilbert space is $\mathscr{H} = \mathscr{L}^{p=2}(-\infty,-1] \cup [1,\infty)$. We then impose on Eq. (28) the following minimal requirements, such that its domain is not too artificially restricted.

i $\hat{H}$ is a symmetric (Hermitian) linear operator;
ii $\hat{H}$ can be applied on all functions of the form

$$g(x,s) = P(x,s)\exp\Big(-\frac{x^2}{2}\Big), \tag{29}$$

where P is a polynomial of x and s. Here, it should be pointed out that $\hat{H} = \hat{T} + \hat{V}$, and from Eq. (26), it can be seen that $\hat{T} = -2i\hbar x\partial_x$, $\hat{V} = -i\hbar$. From (ii), $\hat{V}g(x,s)$ must belong to the Hilbert space $\mathscr{H} = \mathscr{L}^2$ defined over the space $x \geq 1$. This is guaranteed as $\mid -i\hbar \mid \leq \hbar$ where $\hbar$ is the reduced Planck constant or Dirac constant, (Planck's constant multiplied by an imaginary number is strictly bounded, i.e. strictly less than infinity). The domain $\mathscr{D}_{\hat{V}}$ of the potential energy $\hat{V}$ consists of all $\phi \in \mathscr{H}$ for which $\hat{V}\phi \in \mathscr{H}$. As such, $\hat{V}$ is self-adjoint. It is not necessary to specify the domain of Eq. (28), as it is only necessary to admit that Eq. (28) is defined on a certain $\mathscr{D}_{\hat{H}}$ such that (i) and (ii) are satisfied. If we denote by $\mathscr{D}_1$ the set of all functions in Eq. (29), then (ii) implies that $\mathscr{D}_{\hat{H}} \supseteq \mathscr{D}_1$. By letting $\hat{H}_1$ be the contraction of $\hat{H}$ with domain $\mathscr{D}_1$, i.e., $\hat{H}$ is an extension of $\hat{H}_1$, and letting $\tilde{H}_1$ be the closure of $\hat{H}$, it can be seen that $\tilde{H}_1$ is self-adjoint. Since $\hat{H}$ is symmetric and $\hat{H} \supseteq \hat{H}_1$, i.e., $\hat{H}$ is an extension of $\hat{H}_1$, it follows that $\tilde{H} = \tilde{H}_1$ and $\hat{H}$ is essentially self-adjoint, where $\tilde{H}$ is the unique self-adjoint extension [43]. Other than eigenfunctions $\phi_s(x)$ in configuration space as seen in Eq. (23), it is useful to represent eigenfunctions in momentum space $\phi_s(p)$. The transformation between configuration space eigenfunctions and momentum space eigenfunctions can be obtained via Plancherel transforms [45], where the one-to-one correspondence $\phi_s(x) \rightleftharpoons \phi_s(p)$ is linear and isometric.

2.1 Green's function

In order to obtain eigenstates that are orthonormal when $x \neq 1$, as seen in Eq. (24), we begin by writing Eq. (28) as the eigenvalue equation

$$-2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x) = t\phi_s(x). \tag{30}$$

Dividing by $-2i\hbar$ on both sides and rearranging the terms, we obtain

$$\phi_s' + \frac{1}{x}\frac{t}{2i\hbar}\phi_s = -\frac{1}{2x}\phi_s. \tag{31}$$

This can be written as

$$\phi_s' + k^2 = Q, \tag{32}$$

where

$$k \equiv \sqrt{\frac{t}{2i\hbar x}}, \tag{33}$$

and

$$Q \equiv -\frac{1}{2x}\phi_s. \tag{34}$$

Therefore, we can express Eq. (30) as

$$(\partial_x + k^2)\phi_s = Q. \tag{35}$$

In order to solve an inhomogeneous differential equation such as Eq. (35), we can find a Green's function that uses a delta function source, viz.,

$$(\partial_x + k^2)G(x) = \delta(x), \tag{36}$$

where the delta potential is given by [39]

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{37}$$

It then follows from Eq. (36) that we can express ϕ_s as an integral to obtain $Q(x)$, i.e.,

$$\phi_s(x) = \int_{\mathbb{R}^n} G(x - x_0)Q(x_0)d^n x_0, \tag{38}$$

and it must satisfy

$$\begin{aligned} (\partial_x + k^2)\phi_s(x) &= \int_{\mathbb{R}^n} \Big[(\partial_x + k^2)G(x - x_0)\Big] Q(x_0)d^n x_0 \\ &= \int_{\mathbb{R}^n} \delta(x - x_0)Q(x_0)d^n x_0 = Q(x). \end{aligned} \tag{39}$$

In order to obtain the Green's function $G(x)$ such that a solution to Eq. (36) can be obtained, we take the Fourier transform which turns the differential equation into an algebraic one, like

$$G(x) = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)g(\omega)d\omega, \tag{40}$$

where $g(\omega)$ is the projection, and $\exp(i\omega x)$ is the complete basis set. Upon inserting Eq. (40) into Eq. (36), we obtain

$$(\partial_x + k^2)G(x) = \frac{1}{\sqrt{2\pi}} \int g(\omega)(\partial_x + k^2)\exp(i\omega x)d\omega = \delta(x). \tag{41}$$

However, since

$$\partial_x \exp(i\omega x) = i\omega \exp(i\omega x), \tag{42}$$

and

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x) d\omega, \tag{43}$$

Eq. (36) can be expressed as

$$\frac{1}{\sqrt{2\pi}} \int (i\omega + k^2) \exp(i\omega x) g(\omega) d\omega = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x) d\omega, \tag{44}$$

where

$$g(\omega) = \frac{1}{\sqrt{2\pi}(i\omega + k^2)}. \tag{45}$$

Hence we have poles at

$$k = \pm\sqrt{i\omega}. \tag{46}$$

Now consider the contour integral

$$\frac{1}{\sqrt{2\pi}} \int_C f(z) dz = \frac{1}{\sqrt{2\pi}} \int_C \frac{\exp(izx)}{(iz + k^2)} dz. \tag{47}$$

Since $\exp(izx)$ is an entire function, Eq. (47) has singularities only at the poles, as given in Eq. (46), i.e., $z = ik^2$. As $f(z)$ is

$$\frac{\exp(izx)}{(iz + k^2)} = \frac{\exp(izx)}{i} \frac{1}{(z - ik^2)}, \tag{48}$$

the residue of $f(z)$ at $z = ik^2$ is

$$\text{Res}_{z=ik^2} f(z) = \frac{\exp(-k^2 x)}{i}. \tag{49}$$

According to the residue theorem, we then obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_C f(z) dz &= \frac{2\pi i}{\sqrt{2\pi}} \text{Res}_{z=ik^2} f(z) \\ &= \sqrt{2\pi} \exp(-k^2 x) = G(x). \end{aligned} \tag{50}$$

Hence, the most general solution to Eq. (36) is

$$\phi_s(x) = \sqrt{2\pi} \int_{\mathbb{R}^n} \exp(-k^2 x_0) \Big(- \frac{1}{2x_0} \phi_s(x_0) \Big) d^n x_0. \tag{51}$$

From Eq. (23) it can be seen that $\phi_s(x_0) = x_0^{-s}$. As such,

$$\begin{aligned}\phi_s(x) &= -\sqrt{2\pi}\int_{\mathbb{R}^n} \exp(-k^2 x_0)\Big(\frac{x_0^{-s-1}}{2}\Big) d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} \frac{\exp(-k^2 x_0)}{x_0^{s+1}} d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} \frac{\exp(-\frac{tx_0}{2i\hbar x})}{x_0^{s+1}} d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} \cos\Big(\frac{tx_0}{2\hbar x}\Big)\frac{1}{x_0^{s+1}} d^n x_0 - i\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} \sin\Big(\frac{tx_0}{2\hbar x}\Big)\frac{1}{x_0^{s+1}} d^n x_0,\end{aligned} \tag{52}$$

Moreover, by using Eq. (23) it can be seen that

$$\begin{aligned}\int_{\mathbb{R}^n} \cos\Big(\frac{tx_0}{2\hbar x}\Big)\frac{1}{x_0^{s+1}} d^n x_0 &= \int_{\mathbb{R}^n} \cos\Big(\frac{tx_0}{2\hbar x}\Big)\frac{x_0^{-\sigma}}{x_0} \cos\Big(t\ln(x_0)\Big) d^n x_0 \\ &\quad - i\int_{\mathbb{R}^n} \cos\Big(\frac{tx_0}{2\hbar x}\Big)\frac{x_0^{-\sigma}}{x_0} \sin\Big(t\ln(x_0)\Big) d^n x_0,\end{aligned} \tag{53}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} \sin\Big(\frac{tx_0}{2\hbar x}\Big)\frac{1}{x_0^{s+1}} d^n x_0 &= \int_{\mathbb{R}^n} \sin\Big(\frac{tx_0}{2\hbar x}\Big)\frac{x_0^{-\sigma}}{x_0} \sin\Big(t\ln(x_0)\Big) d^n x_0 \\ &\quad + i\int_{\mathbb{R}^n} \sin\Big(\frac{tx_0}{2\hbar x}\Big)\frac{x_0^{-\sigma}}{x_0} \cos\Big(t\ln(x_0)\Big) d^n x_0.\end{aligned} \tag{54}$$

Since $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$, it can be seen that

$$\begin{aligned}\phi_\sigma(x) &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} \cos\Big(\frac{tx_0}{2\hbar x}\Big)\frac{x_0^{-\sigma}}{x_0} \cos\Big(t\ln(x_0)\Big) d^n x_0 \\ &\quad - \sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} \sin\Big(\frac{tx_0}{2\hbar x}\Big)\frac{x_0^{-\sigma}}{x_0} \sin\Big(t\ln(x_0)\Big) d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} x_0^{-\sigma-1} \cos\left(ix_0 k^2 - t\log(x_0)\right) d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n} x_0^{-\sigma-1}\Big[\cosh\left(k^2 x_0\right)\cos\Big(t\log(x_0)\Big) \\ &\quad + i\sinh\left(k^2 x_0\right)\sin\Big(t\log(x_0)\Big)\Big] d^n x_0,\end{aligned} \tag{55}$$

and

$$\begin{aligned}\phi_t(x) &= \sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n}\cos\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\sin\Big(t\ln(x_0)\Big)d^n x_0\\ &-\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n}\sin\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\cos\Big(t\ln(x_0)\Big)d^n x_0\\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n}x_0^{-\sigma-1}\sin\left(ix_0k^2 - t\log(x_0)\right)d^n x_0\\ &= -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^n}x_0^{-\sigma-1}\Big[-\cosh\left(k^2x_0\right)\sin\Big(t\log(x_0)\Big)\\ &+ i\sinh\left(k^2x_0\right)\cos\Big(t\log(x_0)\Big)\Big]d^n x_0. \end{aligned} \tag{56}$$

Here, we can use the identities

$$\cos\Big(t\log(x_0)\Big) = \frac{1}{2}x_0^{-it} + \frac{1}{2}x_0^{it}, \tag{57}$$

and

$$\sin\Big(t\log(x_0)\Big) = \frac{i}{2}x_0^{-it} - \frac{i}{2}x_0^{it}, \tag{58}$$

to rewrite Eqs. (55)-(56) as

$$\begin{aligned}\phi_\sigma(x) &= -\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}x_0^{-\sigma-1}\cos\Big(t\log(x_0)\Big)\exp(-k^2x_0)dx_0\\ &= -\frac{1}{2}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}e^{-k^2x_0}\left(1+x_0^{2it}\right)x_0^{-\sigma-it-1}dx_0, \end{aligned} \tag{59}$$

and

$$\begin{aligned}\phi_t(x) &= \sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}x_0^{-\sigma-1}\sin\Big(t\log(x_0)\Big)\exp(-k^2x_0)dx_0\\ &= -\frac{1}{2}i\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}e^{-k^2x_0}\left(-1+x_0^{2it}\right)x_0^{-\sigma-it-1}dx_0. \end{aligned} \tag{60}$$

Taking $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$, we arrive at the expression using Eq. (33)

$$\begin{aligned}\phi_s(x) &= \sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}\left(-e^{-k^2x_0}\right)x_0^{-\sigma-it-1}dx_0\\ &= \frac{\sqrt{\frac{\pi}{2}}\left(k^*-ik\right)e^{-\frac{1}{2}\pi(t+3i\sigma)}\left(e^{2\pi t}-e^{2i\pi\sigma}\right)\left(-k^4\right)^{\frac{1}{2}(\sigma+it)}\mathit{\Gamma}(-it-\sigma)}{2k^*}\\ &= \frac{\sqrt{\pi}2^{-\sigma-it-\frac{3}{2}}e^{-\frac{1}{2}\pi(t+3i\sigma)}\left(e^{2\pi t}-e^{2i\pi\sigma}\right)\left(t-x\sqrt{\frac{t^2}{x^2}}\right)\mathit{\Gamma}(-it-\sigma)\left(\frac{t^2}{x^2}\right)^{\frac{1}{2}(\sigma+it)}}{t}\\ &= 0\ \forall\ x,t\ \in\mathbb{R}^+. \end{aligned} \tag{61}$$

Remark 1 Since Eq. (61) a trivial solution, from Eq. (31) it can be seen that by taking $y = \phi_s$,

$$y' + \frac{1}{x}\Big(\frac{1}{2} + \frac{t}{2i\hbar}\Big)y = 0, \tag{62}$$

a nontrivial solution is admitted as

$$y = \frac{1}{x^s}, \tag{63}$$

where

$$\boxed{s = \frac{1}{2} + \frac{t}{2i\hbar}.} \tag{64}$$

2.2 Integrability

Theorem 1 *The eigenstate* $\phi_s(x) = x^{-s}: \ \mathbb{X} \to \mathbb{C}$ *is measurable. That is,* $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$ *where* $\phi_\sigma, \phi_t: \ \mathbb{E} \to (-\infty, -1] \cup [1, \infty)$ *are measurable for* $s = \sigma + it = |s|\exp(i\theta)$*, and* $|s| = \sqrt{\sigma^2 + t^2}$*,* $\theta = \arctan(t/\sigma)$ *and* $\sigma, t \in \mathbb{R}$*.*

Proof Owing to the one-to-one correspondence obtained from Plancherel transforms between configuration space and momentum space eigenstates, it can be seen that

$$\begin{aligned}\phi_s(p) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_s(x)\exp(-ipx)dx\\ &= \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}i\pi s\Big)(\mathrm{sgn}(p)+1)\sin(\pi s)\Gamma(1-s)\,|p|^{s-1}\\ &= \frac{i}{\sqrt{2\pi}}\Big(\mathrm{sgn}(p)+1\Big)e^{\frac{1}{2}\pi(t-i\sigma)}\sinh\Big(\pi(t-i\sigma)\Big)\Gamma(-it-\sigma+1)\,|p|^{\sigma+it-1},\\ &\quad 0<\sigma<1.\end{aligned} \tag{65}$$

and

$$\phi_s(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_s(p)\exp(ipx)dp. \tag{66}$$

Since

$$\begin{aligned}\parallel \phi_s(x) \parallel_1 &\equiv \int_{-\infty}^{-1}|\phi_s(x)|dx + \int_{-1}^{1}|\phi_s(x)|\delta(x)dx + \int_{1}^{\infty}|\phi_s(x)|dx\\ &= \int_{-\infty}^{-1}|\phi_s(p)|dp + \int_{-1}^{1}|\phi_s(p)|\delta(p)dp + \int_{1}^{\infty}|\phi_s(p)|dp \equiv \parallel \phi_s(p) \parallel_1,\end{aligned} \tag{67}$$

from which

$$\parallel \phi_s(x) \parallel_1 = \parallel \phi_s(p) \parallel_1 = -\frac{1}{s\pi^{1/2}}\exp\Big(\frac{1}{2}\pi\Im(s)\Big)\sqrt{\sin^2(\pi s)}\sqrt{\Gamma(1-s)^2}. \tag{68}$$

It then follows that ϕ_s is complex square-integrable, i.e.,

$$\phi_s(x) \in \mathscr{H} \iff \int_{\mathbb{E}} |\phi_s(x)| d\mu < +\infty. \tag{69}$$

Theorem 2 *Let the complex valued eigenstate* $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$ *where* $s = \sigma + it = |s| \exp(i\theta)$, *and* $|s| = \sqrt{\sigma^2 + t^2}$, $\theta = \arctan(t/\sigma)$, *and let the measurable subset* $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$. *The* $\mathscr{H} = \mathscr{L}^2$*-norm of the complex-valued eigenstate* $\phi_s = x^{-s}$ *is* ∞, *i.e.,* ϕ_s *is not* $p = 2$ *integrable at* $\sigma = 1/2$.

Proof Owing to the one-to-one correspondence obtained from Plancherel transforms between configuration space and momentum space eigenstates, it can be seen that

$$\begin{aligned}
\phi_s(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(x) \exp(-ipx) dx \\
&= \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2} i\pi s\Big)\Big(\mathrm{sgn}(p) + 1\Big) \sin(\pi s) \Gamma(1-s)\, |p|^{s-1} \\
&= \frac{i}{\sqrt{2\pi}} \Big(\mathrm{sgn}(p) + 1\Big) e^{\frac{1}{2}\pi(t - i\sigma)} \sinh\Big(\pi(t - i\sigma)\Big) \Gamma(-it - \sigma + 1)\, |p|^{\sigma + it - 1}, \\
&\quad 0 < \sigma < 1.
\end{aligned} \tag{70}$$

and

$$\phi_s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(p) \exp(ipx) dp, \tag{71}$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \cos\Big(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\Big), \tag{72}$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \sin\Big(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\Big) \tag{73}$$

for $x \in \mathbb{R}^+_{\geq 1}$. Since

$$\| \phi_s \|_p = \Big(\int_1^\infty | \phi_s(x) |^p \, dx \Big)^{\frac{1}{p}}, \tag{74}$$

and

$$\| \phi_s(p) \|_p = \Big(\int_1^\infty | \phi_s(p) |^p \, dp \Big)^{\frac{1}{p}}, \tag{75}$$

Here, the reader is cautioned not to confuse the $\mathscr{L}^p$-norm with the momentum p. It can easily be seen that the $\mathscr{L}^p$-norm of ϕ_s is also of indeterminant form for $x \in (-\infty, -1]$. The $\mathscr{L}^p$-norm vanishes for $x \in [-1, 1]$ owing to the Dirac delta (singularity) at the origin $x = 0$ [39].

from which

$$\| \phi_s(p) \|_p = \| \phi_s(x) \|_p = \left(\frac{1}{p\sigma - 1} \right)^{\frac{1}{p}}. \tag{76}$$

It then follows that as $\sigma \to 1/2$,

$$\| \phi_s(p) \|_p = \| \phi_s(x) \|_p = \left(\frac{1}{\frac{p}{2} - 1} \right)^{\frac{1}{p}}, \tag{77}$$

such that the $\mathscr{L}^{p=2}$-norm of ϕ_s is of indeterminant form. Furthermore, it can be seen from

$$\lim_{p \to 2} \left(\frac{1}{\frac{p}{2} - 1} \right)^{\frac{1}{p}}, \tag{78}$$

and letting

$$y = \left(\frac{1}{\frac{p}{2} - 1} \right)^{\frac{1}{p}}, \tag{79}$$

then

$$\begin{aligned} \ln(y) &= \frac{1}{p} \ln \left(\frac{1}{\frac{p}{2} - 1} \right) \\ &= \frac{1}{p} \Big(\ln(1) - \ln \Big(\frac{p}{2} - 1 \Big) \Big) \\ &= -\frac{1}{p} \ln \Big(\frac{p}{2} - 1 \Big), \end{aligned} \tag{80}$$

and

$$\begin{aligned} \lim_{p \to 2} \ln(y) &= \lim_{p \to 2} \Big(-\frac{1}{p} \ln \Big(\frac{p}{2} - 1 \Big) \Big) \\ &= \infty. \end{aligned} \tag{81}$$

Exponentiating both sides, we obtain

$$\begin{aligned} \exp \Big[\lim_{p \to 2} \ln(y) \Big] &= \lim_{p \to 2} \Big[\exp \Big(\ln(y) \Big) \Big] \\ &= \lim_{p \to 2} y = \exp(\infty) = \infty, \end{aligned} \tag{82}$$

such that we obtain the infinite density [28]

$$\| \phi_s(p) \|_{p=2} = \| \phi_s(x) \|_{p=2} = \infty. \tag{83}$$

Corollary 1 *Consider the Hamiltonian observable given by*

$$\hat{H}\phi_s(x) = -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x). \tag{84}$$

Although the action of $\hat{H}$ is in principle well-defined for all $\phi_s(x) \in \mathscr{L}^2$, there are functions which are in $\mathscr{L}^2$, but for which $\hat{H}\phi_s(x)$ is no longer an element of $\mathscr{L}^2$, e.g., when $\sigma = 1/2$,

$$\begin{aligned}\phi_{\frac{1}{2}+it}(x) = &\frac{e^{t\arg(x)}\cos\left(\frac{\arg(x)}{2} + \frac{1}{2}t\log\left(x^2\right)\right)}{\sqrt[4]{x^2}} \\ &- \frac{ie^{t\arg(x)}\sin\left(\frac{\arg(x)}{2} + \frac{1}{2}t\log\left(x^2\right)\right)}{\sqrt[4]{x^2}}.\end{aligned} \tag{85}$$

Therefore the domain of $\hat{H}$ is given by

$$\mathscr{D}(\hat{H}) = \left\{\phi_s(x) \in \mathscr{L}^2 : \int_{-\infty}^{\infty} \Big| -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\Big|^2 \delta(x-1)dx < \infty\right\} \subset \mathscr{L}^2. \tag{86}$$

Similarly, the domain of $\hat{H}^2$ is

$$\begin{aligned}\mathscr{D}(\hat{H}^2) = &\left\{\phi_s(x) \in \mathscr{L}^2 : \int_{-\infty}^{\infty} \Big|\Big(-2i\hbar\sqrt{x}\partial_x\sqrt{x}\Big)^2\phi_s(x)\Big|^2 \delta(x-1)dx < \infty\right\} \\ &\subset \mathscr{D}(\hat{H}),\end{aligned} \tag{87}$$

etc. As such, we define the dense subspace of $\mathscr{H}$ as

$$\Phi \equiv \bigcap_{n=0}^{\infty} \mathscr{D}(\hat{H}^n), \tag{88}$$

such that for every $\phi_s(x) \in \Phi$, the solution is well-defined at $\sigma = 1/2$.

Eqs. (65) and (66) are two vector representations of the same Hilbert space $\mathscr{H} = \mathscr{L}^{p=2}(-\infty, -1] \cup [1, \infty)$. From Eq. (26), it can be seen that

$$\hat{T} = -2i\hbar x\partial_x, \tag{89}$$

such that we define a multiplicative operator $\hat{T}_0$ in momentum space i.e. $(\hat{T}_0\Phi_s)(p) = \hat{T}_0(p)\Phi_s(p)$, where

$$\hat{T}_0(p) = 2\hat{x}\hat{p}. \tag{90}$$

Here, it should be pointed out that as $\hat{x} = i\hbar d/dp$, as such Eq. (90) reduces to

$$\hat{T}_0(p) = 2i\hbar, \tag{91}$$

and Eq. (26) is then rewritten in momentum space as $\hat{H}(p) = i\hbar$. The domain $\mathscr{D}_0$ of $\hat{T}_0$ is defined as the set of all functions $\phi_s(p) \in \mathscr{H}$ such that $\hat{T}_0(p)\phi_s(p) \in$

$\mathscr{H}$. As such, $\hat{T}_0$ is definitively self-adjoint. From Eq. (29) we have defined the set $\mathscr{D}_1$ of functions in configuration space. From the Plancherel transform [45] of Eq. (29), we obtain the set $\mathscr{D}_1$ of functions in momentum space having the form

$$G(p,s) = P(p,s)\exp\Big(-\frac{p^2}{2}\Big), \tag{92}$$

where P is a polynomial of p and s. Eqs. (65) and (66) are true if $\phi_s(x) \in \mathscr{D}_1$ or $\phi_s(p) \in \mathscr{D}_1$ and since $\phi_s(p) \in \mathscr{D}_1 \to 0$ as $p \to \infty$ then $\mathscr{D}_1 \subseteq \mathscr{D}_0$. Moreover, for $\phi_s(x) \in \mathscr{D}_1$, $\hat{T}_0$ coincides with Eq. (89) [43]. Using Eq. (65) and $\hat{H}(p) = i\hbar$, the eigenrelation

$$\hat{H}(p)\,|\Phi_s(p)\rangle = \lambda\,|\Phi_s(p)\rangle \tag{93}$$

is obtained. In order to find the expectation value for $\hat{H}$ we take the complex conjugate of Eq. (93), set $\hbar = 1$, multiply by the eigenfunction $\phi_s(p)$, and then integrate over p to obtain

$$\int_{-\infty}^{\infty}\Big(i\frac{e^{-\frac{1}{2}i\pi s}(\mathrm{sgn}(p)+1)\sin(\pi s)\Gamma(1-s)\,|p|^{s-1}}{2\pi^{1/2}}\Big)^* \\ \Big(\frac{e^{-\frac{1}{2}i\pi s}(\mathrm{sgn}(p)+1)\sin(\pi s)\Gamma(1-s)\,|p|^{s-1}}{2\pi^{1/2}}\Big)dp = \lambda^* \parallel \Phi_s \parallel_p, \tag{94}$$

where λ is the eigenvalue.

2.3 Bender-Brody-Müller-Schrödinger Equation

Definition 21 *The BBM-Schrödinger equation is [44]*

$$-\hbar\partial_s\,|\Psi_s(x)\rangle = i\,\Big[\hat{\Delta}^{-1}\hat{x}\hat{p}\hat{\Delta} + \hat{\Delta}^{-1}\hat{p}\hat{x}\hat{\Delta}\Big]\,|\Psi_s(x)\rangle\,, \tag{95}$$

where $\hat{\Delta} = 1 - \exp(-\partial_x)$, $\hat{x} = x$, $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, $x \in \mathbb{R}^+ \geq 1$ *owing to the difference operator* $\hat{\Delta}\,|\Psi_s(x)\rangle$, *and* $s \in \mathbb{C}$.

Upon inserting Eq. (20) into Eq. (95) for $x \in \mathbb{R}^+$, we obtain the symmetrized BBM-Schrödinger equation, i.e.,

$$\begin{aligned}\partial_s\,|\phi_s(x)\rangle &= 1/2(\partial_\sigma - i\partial_t)\,|\phi_s(x)\rangle \\ &= -\frac{2}{\hbar}\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle\,.\end{aligned} \tag{96}$$

Theorem 3 *Let the complex-valued eigenstate*

$$\phi_s(x) = \frac{1}{x^{\frac{1}{2}+\frac{t}{2i\hbar}}}, \tag{97}$$

where $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2+t^2}$, $\theta = \arctan(t/\sigma)$ *and* $\sigma, t \in \mathbb{R}$. *The eigenvalues are real for the expectation value of the Hamiltonian operator* $\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}$.

Proof Let $|\phi_s(x)\rangle$ be an eigenstate of $\hat{H}$ with eigenvalue t, i.e.,

$$\hat{H}\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{98}$$

In order to find the expectation value of $\hat{H}$ we multiply $\hat{H}$ by the eigenstate, take the complex conjugate, and then multiply the result by the eigenstate and integrate to obtain

$$\begin{aligned}2i\int_{-\infty}^{\infty}\left(\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\right)^*\phi_s(x)\delta(x-1)dx &= t^*\int_{-\infty}^{\infty}\phi_s^*(x)\phi_s(x)\delta(x-1)dx\\ &= t^*\parallel\phi\parallel\,.\end{aligned} \tag{99}$$

Integrating by parts on the LHS then gives

$$t = t^*. \tag{100}$$

Theorem 4 *Let the complex-valued eigenstate* $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$ *where* $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2+t^2}$, $\theta = \arctan(t/\sigma)$ *and* $\sigma, t \in \mathbb{R}$. *For the Hamiltonian operator* $\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}$, *all of the eigenvalues* t *occur at* $\sigma = 1/2$.

Proof Let $|\phi_s(x)\rangle$ be an eigenstate of $\hat{H}$ with eigenvalue t, i.e.,

$$\hat{H}\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{101}$$

In order to find the expectation value of $\hat{H}$ we multiply $\hat{H}$ by the eigenstate, take the complex conjugate, and then multiply the result by the eigenstate and integrate over $\mathbb{E}$ to obtain

$$\begin{aligned}2i\int_{-\infty}^{\infty}\left(\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\right)^*\phi_s(x)\delta(x-1)dx &= t^*\int_{-\infty}^{\infty}\phi_s^*(x)\phi_s(x)\delta(x-1)dx\\ &= t^*\parallel\phi\parallel\,.\end{aligned} \tag{102}$$

Integrating by parts on the LHS then gives

$$-2t + i(1-2\sigma) = t^*. \tag{103}$$

Since from Eq. (100) we have $t = t^*$, it can therefore be seen that

$$\boxed{\sigma = \frac{1}{2}\;\forall\, t.} \tag{104}$$

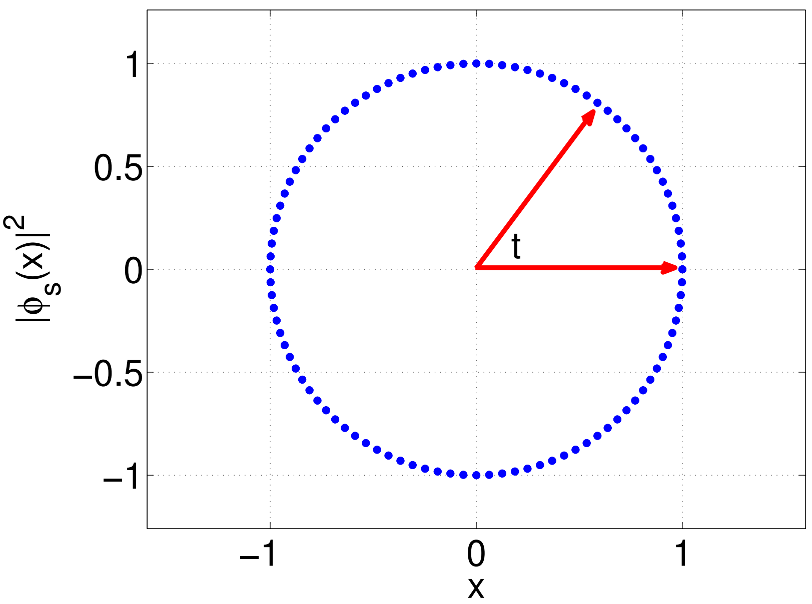

Fig. 1 From Eq. (105), the density $|\phi_s(x)|^2$, where $s = |s|\exp(i\arctan(t/\sigma)) = 1/2 - \log(x)/2 = 1/2 + t/2i\hbar$. Parity symmetry is exhibited about the origin, as $\langle \Pi \rangle = \pi W(0,0)/2$ [47]. The density is normalized when $x\cos(t) = 1$ (color online).

2.4 Convergence

Theorem 5 *For the symmetrized BBM-Schrödinger equation, i.e.,* $\hbar\partial_s |\phi_s(x)\rangle = -2\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle$*, the complex-valued eigenstate* $|\phi_s(x)\rangle = x^{-s}$ *where* $s = \sigma + it = |s|\exp(i\theta)$*,* $|s| = \sqrt{\sigma^2 + t^2}$*,* $\theta = \arctan(t/\sigma)$ *and* $\sigma, t \in \mathbb{R}$ *normalizes at* $x\cos(t) = 1$*, i.e., the density* $|\phi_s(x)|^2 = 1$*.*

Proof In order to obtain convergent solutions to the unsymmetric BBM-Schrödinger Eq. (95), it can be seen that upon inserting Eq. (20) into the symmetric Eq. (96), we obtain

$$s = |s|\exp\Big(i\arctan(t/\sigma)\Big) = \frac{1}{2} - \frac{\log(x)}{2}. \tag{105}$$

Hence at $x = 1$,

$$\sigma = \frac{1}{2} - it \tag{106}$$

where $t \in \mathbb{R}$. This condition is required such that the density is normalized in agreement with Eq. (88), i.e.,

$$\begin{aligned}\parallel \phi_s \parallel_2 &= \sum_m \sum_n \hat{b}_n(s)\hat{b}^\dagger_m(s) \left\langle \phi_m | \phi_n \right\rangle \\ &= \sum_n |\hat{b}_n(s)|^2 \\ &= 1. \end{aligned} \tag{107}$$

Here it should be pointed out that by taking Eqs. (64) and (105) and inserting them into Eq. (23) gives the eigenequation relation

$$\frac{1}{x^{\frac{1}{2}-\frac{\log(x)}{2}}} = \frac{1}{x^{\frac{1}{2}+\frac{t}{2i\hbar}}}. \tag{108}$$

Hence we obtain the eigenfunction

$$\begin{aligned}\phi_s(x) &= \frac{1}{x^s} \\ &= (e^{\frac{it}{\hbar}})^{\frac{1}{2}[-1+\log(e^{\frac{it}{\hbar}})]}. \end{aligned} \tag{109}$$

Theorem 6 *For the BBM equation [8, 9], i.e.,*

$$\frac{1}{1-e^{-i\hat{p}}}(\hat{x}\hat{p}+\hat{p}\hat{x})(1-e^{-i\hat{p}}) \left|\Psi_s(x)\right\rangle = t \left|\Psi_s(x)\right\rangle, \tag{110}$$

the eigenvalues of the analytic continuation of the eigenfunction at $|\sigma| = 1/2$, *are periodically* ($2\pi n$) *zero in the complex plane* $\forall\ n \in \mathbb{Z}$.

Proof At $x = \sec(t = 2\pi n) = 1$, the normalization constraint Eq. (107) is satisfied, $\sigma = \frac{1}{2} - it$, and Eq. (17) can be written

$$\begin{aligned}\Psi_s(x=1) &= -\zeta(s=1/2, 2) \\ &= -\Gamma(1/2)\frac{1}{2\pi i}\oint_C \frac{\sqrt{z}e^{2z}}{1-e^z}dz \\ &= 1-\zeta(\sigma = \frac{1}{2} - it). \end{aligned} \tag{111}$$

where the contour C is about $\mathbb{R}^-$. From the analytic continuation relations of Eq. (1)

$$\begin{aligned}\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} &= \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\Big(-i\cdot t\ln(n)\Big)}{n^\sigma} \\ &= \frac{1}{1-2^{1-s}}\Big[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\Big(t\cdot\ln(n)\Big)}{n^\sigma} \\ &\quad - i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\Big(t\cdot\ln(n)\Big)}{n^\sigma}\Big], \end{aligned} \tag{112}$$

$$1-\Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^{*}=1-\frac{1}{1-2^{1-s^*}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\Big(i\cdot t\ln(n)\Big)}{n^{\sigma}}$$
$$=1-\frac{1}{1-2^{1-s^*}}\Big[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\Big(t\cdot\ln(n)\Big)}{n^{\sigma}}$$
$$+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\Big(t\cdot\ln(n)\Big)}{n^{\sigma}}\Big]. \tag{113}$$

$$\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{2^{-\sigma+1}\cos\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}, \tag{114}$$

such that

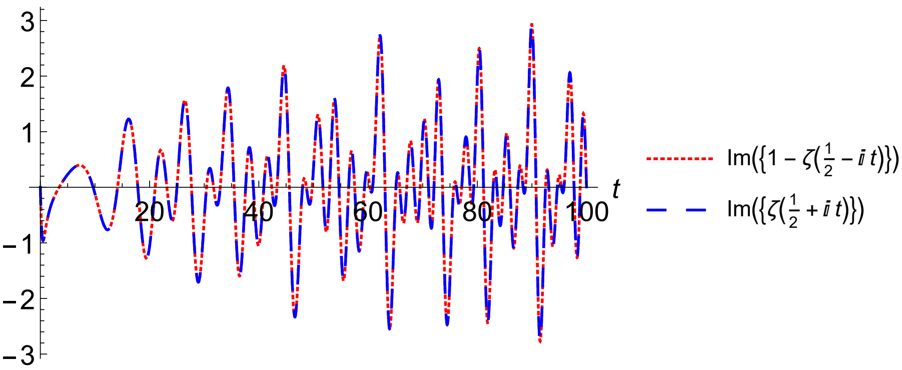

Fig. 2 Plot of the imaginary components of Eq. (1). Results are compared with Eq. (116) (color online).

$$
\begin{aligned}
1-&\Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^*\\
&=1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{2^{-\sigma+1}\cos\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}\\
&+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}\\
&+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}\\
&+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}\\
&+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{2^{-\sigma+1}\cos\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}\\
&+i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cdot\frac{-\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2},
\end{aligned}
\tag{115}
$$

Owing to the periodicity of $t = 2\pi n$ at $x = \sec(t)$, i.e. Eq. (106), it can be seen that

$$\Im\Big[\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big] = \Im\Big[1 - \Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^*\Big]. \tag{116}$$

Owing to Eq. (104), at $|\sigma| = 1/2$ we obtain

$$\Im\Big[\zeta(s)\Big] = \quad i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\cdot\frac{\sin\Big(t\ln(n)\Big) - \sqrt{2}\sin\big(t\log\big(\frac{n}{2}\big)\big)}{2\sqrt{2}\cos\Big(t\log(2)\Big) - 3}. \tag{117}$$

However, since at $|\sigma| = 1/2$ the eigenvalues $t = n$ are not observable, i.e., $\langle\hat{H}\rangle = t = 2\pi n$ in the complex plane, we have

$$\begin{aligned}\Im\Big[\zeta(s)\Big] &= i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\cdot\frac{\sin\Big(\cancelto{0}{2\pi n}\ln(n)\Big) - \sqrt{2}\sin\Big(\cancelto{0}{2\pi n}\log\big(\frac{n}{2}\big)\Big)}{2\sqrt{2}\cos\Big(\cancelto{0}{2\pi n}\log(2)\Big) - 3}\\ &= 0\ \forall\ n\in\mathbb{Z}.\end{aligned} \tag{118}$$

Remark 2 It has been noted that there is a uniquely defined relation between prime numbers and the imaginary parts of Eq. (1), independent of their real part [12,48].

Remark 3 In the theory of computation, an observable is called *computable*, or *effective*, if and only if its behavior is given by a computable function [32]. From Theorem 6, it can be seen that the *BBM conjecture is not analytically computable*. If the BBM conjecture is uncomputable, there is no proof it is false. If we find an eigenvalue, that is a proof that it is false. Thus if the BBM conjecture is uncomputable there are no eigenvalues [49].

Remark 4 The second axiom of Kolmogorov (unit measure) states that the probability that at least one of the elementary events in the entire sample space will occur is 1. [50].

2.5 Second Quantization

Theorem 7 *By representing the complex-valued eigenstate* $|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i\,|\phi_t(x)\rangle = x^{-s}$ *where* $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2+t^2}$, $\theta = \arctan(t/\sigma)$ *and* $\sigma, t \in \mathbb{R}$ *as a linear combination of basis states, then the eigenvalues of the Hamiltonian operator* $-2i\hbar\sqrt{x}\partial_x\sqrt{x}$ *are real numbers.*

Proof A standard way to introduce topology into the algebra of observables is to make them operators on a Hilbert space. In order to perform a second quantization [51], we can express the complex-valued eigenstate as a linear combination of basis states

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)\,|\phi_n(x)\rangle\,, \tag{119}$$

where $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2 + t^2}$, $\theta = \arctan(t/\sigma)$, $s \in \mathbb{C}$, and $\sigma, t \in \mathbb{R}$. As such, using Eqs. (23) and (104) we can rewrite Eq. (119) as

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}} \hat{b}_n(s) x^{-\frac{1}{2}-\frac{n}{2i\hbar}}. \tag{120}$$

From using this second quantization in Eq. (96), we find

$$\hbar \frac{d}{ds}\hat{b}_n(s) = t_n \hat{b}_n(s). \tag{121}$$

We now find a Hamiltonian that yields Eq. (121) as the equation of motion, hence, we take

$$\langle\phi_{s'}(x)|\,\hat{H}\,|\phi_s(x)\rangle = -2\int_{-\infty}^{\infty} \langle\phi_{s'}(x)|\,\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle\,\delta(x-1)dx, \tag{122}$$

as the expectation value. Upon substituting Eq. (120) into Eq. (122), we obtain the harmonic oscillator

$$\begin{aligned}
\langle\phi_m(x)|\,\hat{H}\,|\phi_n(x)\rangle &= -2i\hbar \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{x^{\frac{1}{2}-\frac{m}{2i\hbar}}}\sqrt{x}\partial_x\sqrt{x}\,\frac{1}{x^{\frac{1}{2}+\frac{n}{2i\hbar}}}\delta(x-1)dx \\
&= -2i\hbar \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{inx^{-\frac{im}{2\hbar}+\frac{in}{2\hbar}-1}}{2\hbar}\delta(x-1)dx \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \hat{b}_m^\dagger(s)\hat{b}_n(s)\,\langle m|\,n\,|n\rangle\,,
\end{aligned} \tag{123}$$

for $|m\rangle, |n\rangle = 1, 2, 3, \ldots, \infty$. Hence at $m = n$, $\langle n|n\rangle = \delta_{nn} = 1$ and

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle = \sum_{n\in\mathbb{Z}} |\hat{b}_n(s)|^2 n. \tag{124}$$

Taking $\hat{b}_n(s)$ as an operator, and $\hat{b}_n^\dagger(s)$ as the adjoint, we obtain the usual properties:

$$\begin{aligned}
&[\hat{b}_n(s), \hat{b}_m(s)] = [\hat{b}_n^\dagger(s), \hat{b}_m^\dagger(s)] = 0, \\
&[\hat{b}_n(s), \hat{b}_m^\dagger(s)] = \delta_{nm}.
\end{aligned} \tag{125}$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned}
\hbar\frac{d}{ds}\sum_{n\in\mathbb{Z}}\hat{b}_n(s) &= [\hat{b}_n(s),\hat{H}]_- \\
&= \sum_{m\in\mathbb{Z}} t_m\Big(\hat{b}_n(s)\hat{b}^\dagger_m(s)\hat{b}_m(s) - \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\
&= \sum_{m\in\mathbb{Z}} t_m\Big(\delta_{nm}\hat{b}_m(s) - \hat{b}^\dagger_m(s)\hat{b}_n(s)\hat{b}_m(s) - \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\
&= \sum_{m\in\mathbb{Z}} t_m\Big(\delta_{nm}\hat{b}_m(s) + \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s) - \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\
&= \sum_{n\in\mathbb{Z}} \hat{b}^\dagger_m(s)\hat{b}_n(s)t_n \\
&= \sum_{n\in\mathbb{Z}} \hat{b}^\dagger_m(s)\hat{b}_n(s)n.
\end{aligned} \tag{126}$$

2.6 Holomorphicity

Theorem 8 *The densely defined Hamiltonian operator* $\hat{H} = -2i\sqrt{x}\partial_x\sqrt{x}$ *on the dense subspace* Φ *is symmetric (Hermitian) [52], for the complex-valued eigenstate* $|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i\,|\phi_t(x)\rangle = x^{-s}$ *where* $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2 + t^2}$, $\theta = \arctan(t/\sigma)$ *and* $\sigma, t \in \mathbb{R}$.

Proof By expressing the complex-valued eigenstate as a linear combination of basis states such that

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)\,|\phi_n(x)\rangle\,, \tag{127}$$

where $s\in\mathbb{C}$, $s = \sigma + it = |s|\exp(i\theta)$, $|s| = \sqrt{\sigma^2+t^2}$, $\theta = \arctan(t/\sigma)$, and $\sigma, t\in\mathbb{R}$, it can be seen that by using Eq. (23) we can rewrite Eq. (127) as

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)x^{-\frac{1}{2}-\frac{n}{2i\hbar}}. \tag{128}$$

By taking the inner product

$$\begin{aligned}
(\hat{H}\phi^*_n,\phi_m) &= 2i\hbar\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{1}{x^{\frac{1}{2}+\frac{m}{2i\hbar}}}\sqrt{x}\partial_x\sqrt{x}\;\frac{1}{x^{\frac{1}{2}-\frac{n}{2i\hbar}}}\delta(x-1)dx \\
&= 2i\hbar\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\hat{b}^\dagger_m(s)\hat{b}_n(s)\,\langle m|\int_{-\infty}^{\infty}\Big(-\frac{inx^{\frac{im}{2\hbar}-\frac{in}{2\hbar}-1}}{2\hbar}\Big)\delta(x-1)dx\,|n\rangle \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\hat{b}^\dagger_m(s)\hat{b}_n(s)\,\langle m|\,n\,|n\rangle\,,
\end{aligned} \tag{129}$$

for $|m\rangle, |n\rangle = 1, 2, 3, \ldots, \infty$. Hence at $m = n$, $\langle n|n\rangle = \delta_{nn} = 1$ and

$$\langle \phi_n(x)| \hat{H} |\phi_n(x)\rangle = \sum_{n\in\mathbb{Z}} |\hat{b}_n(s)|^2 n. \tag{130}$$

Furthermore, by taking the inner product

$$\begin{aligned}(\phi_m^*, \hat{H}\phi_n) &= -2i\hbar \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{x^{\frac{1}{2}-\frac{m}{2i\hbar}}} \sqrt{x}\partial_x\sqrt{x}\, \frac{1}{x^{\frac{1}{2}+\frac{n}{2i\hbar}}}\delta(x-1)dx \\ &= -2i\hbar \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \hat{b}_m^\dagger(s)\hat{b}_n(s)\langle m| \int_{-\infty}^{\infty} \frac{inx^{-\frac{im}{2\hbar}+\frac{in}{2\hbar}-1}}{2\hbar}\delta(x-1)dx\,|n\rangle \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \hat{b}_m^\dagger(s)\hat{b}_n(s)\langle m|\, n\, |n\rangle, \end{aligned} \tag{131}$$

for $|m\rangle, |n\rangle = 1, 2, 3, \ldots, \infty$. Hence at $m = n$, $\langle n|n\rangle = \delta_{nn} = 1$ and

$$\langle \phi_n(x)| \hat{H} |\phi_n(x)\rangle = \sum_{n\in\mathbb{Z}} |\hat{b}_n(s)|^2 n. \tag{132}$$

Finally,

$$(\hat{H}\phi_n^*, \phi_m) = (\phi_m^*, \hat{H}\phi_n) = n \in \mathbb{Z} \tag{133}$$

such that $t = 2\pi n$ in the complex plane $\mathbb{C}$.

2.7 Hamilton's Equations

Hamilton's equations in classical mechanics are analogous to quantum mechanics. Here, let us suppose that we have a set of basis states $\{|s\rangle\}$, which are not necessarily eigenstates of the Hamiltonian. For brevity, we assume the basis states are discrete and orthonormal, i.e.

$$\langle s'|s\rangle = \delta_{ss'}. \tag{134}$$

Here we note that the basis states are assumed to be independent of the complex variable $s \in \mathbb{C}$, where $s = \sigma + it$ and $\sigma, t \in \mathbb{R}$. Furthermore, we note that the Hamiltonian Eq. (9) is also independent of s. The instantaneous state of the system at the point s in the complex plane $|\phi_s(x)\rangle$ can be expanded in terms of these basis states, viz. Eq. (119), where

$$\hat{b}_n(s) = \langle \phi_n(x)|\phi_s(x)\rangle = \langle s|\phi_s(x)\rangle. \tag{135}$$

Here, we can treat the complex coefficients $\hat{b}_n(s)$ as coordinates which specify the state of the system, similarly to how position x and momentum p specify the state of a classical system. Like the classical coordinates x and p, the quantum coordinates $\hat{b}_n(s)$ and $\hat{b}_n^\dagger(s)$ are not constant in s, and their s dependence describes the entire system dependency on s. The expectation value of the Hamiltonian using Eq. (119) is

$$\langle H(s)\rangle = \langle \phi_{s'}(x)|H|\phi_s(x)\rangle = \sum_{m,n} \hat{b}_n^\dagger(s)\hat{b}_m(s)\,\langle n|H|m\rangle\,. \tag{136}$$

Each $\hat{b}_n(s)$ actually correspond to two independent degrees of freedom, i.e. their real part and their imaginary part. However, it is of more convenient benefit to treat these two degrees of freedom as the independent variables (generalized coordinates) $\hat{b}_n(s)$ and $\hat{b}_n^\dagger(s)$. As such, we calculate the partial derivative

$$\frac{\partial\langle H\rangle}{\partial\hat{b}_m^\dagger} = \sum_n \hat{b}_n\,\langle m|H|n\rangle = \langle m|H|\phi_s(x)\rangle\,. \tag{137}$$

Then applying Eq. (96) and exploiting the orthonormality of the basis states, this further simplifies to

$$\frac{\partial\langle H\rangle}{\partial\hat{b}_m^\dagger} = \frac{\partial}{\partial s}\hat{b}_m(s). \tag{138}$$

In a similar fashion, we obtain

$$\frac{\partial\langle H\rangle}{\partial\hat{b}_n} = -\frac{\partial}{\partial s}\hat{b}_n^\dagger(s). \tag{139}$$

If we now define *conjugate momentum* variables π_n by

$$\pi_n(s) = \hat{b}_n^\dagger(s) \tag{140}$$

then we obtain Hamilton's equations,

$$\boxed{\frac{\partial\langle H\rangle}{\partial\pi_n} = \frac{\partial}{\partial s}\hat{b}_n(s); \quad \frac{\partial\langle H\rangle}{\partial\hat{b}_n(s)} = -\frac{\partial}{\partial s}\pi_n(s).} \tag{141}$$

3 Similarity Solutions

Since Eq. (96), the BBM-Schrödinger (BBMS) equation possesses symmetry about the origin $x = 0$, we then seek a similarity solution [53] of the form:

$$\phi_s(x) = x^\alpha f(\eta), \tag{142}$$

where $\eta = s/x^\beta$, and the RZSE becomes an ordinary differential equation (ODE) for f. As such, we consider Eq. (96), and introduce the transformation $\xi = \epsilon^a x$, and $\tau = \epsilon^b s$, so that

$$w(\xi,\tau) = \epsilon^c\phi(\epsilon^{-a}\xi, \epsilon^{-b}\tau), \tag{143}$$

where $\epsilon \in \mathbb{R}$, and $\tau \in \mathbb{C}$. From performing this change of variable we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\phi &= \epsilon^{-c}\frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial s} \\ &= \epsilon^{b-c}\frac{\partial w}{\partial \tau},\end{aligned} \tag{144}$$

and

$$\begin{aligned}-2\sqrt{x}\frac{\partial}{\partial x}\sqrt{x}\phi &= -2\sqrt{x}\Big(\frac{\partial\sqrt{x}}{\partial x}\phi + \sqrt{x}\frac{\partial\phi}{\partial x}\Big) \\ &= -2\sqrt{x}\frac{1}{2\sqrt{x}}\phi - 2\sqrt{x}\sqrt{x}\frac{\partial\phi}{\partial x} \\ &= -\phi - 2x\frac{\partial\phi}{\partial x},\end{aligned} \tag{145}$$

where

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \epsilon^{-c}\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} \\ &= \epsilon^{a-c}\frac{\partial w}{\partial \xi}.\end{aligned} \tag{146}$$

By using Eqs. (144)-(146) in Eq. (96), the RZSE is then written

$$\epsilon^{-c}\Big[\epsilon^{b}\frac{\partial w}{\partial \tau} + w + 2\xi\frac{\partial w}{\partial \xi}\Big] = 0, \tag{147}$$

and is invariant under the transformation $\forall\, \epsilon$ if $\epsilon^b = 2$, i.e.,

$$\epsilon^{-c}\Big[\frac{\epsilon^b}{2}\Big(\frac{\partial w}{\partial \tau_{\Re}} - i\frac{\partial w}{\partial \tau_{\Im}}\Big) + w + 2\xi\frac{\partial w}{\partial \xi}\Big] = 0, \tag{148}$$

and

$$b = \frac{\log(2) + 2i\pi n}{\log(\epsilon)},\ \forall\, n \in \mathbb{Z}. \tag{149}$$

Therefore, it can be seen that since ϕ solves the RZSE for x and s, then $w = \epsilon^{-c}\phi$ solves the RZSE at $x = \epsilon^{-a}\xi$, and $s = \epsilon^{-b}\tau$. We now construct a group of independent variables such that

$$\begin{aligned}\frac{\xi}{\tau^{a/b}} &= \frac{\epsilon^a x}{(\epsilon^b s)^{a/b}} \\ &= \frac{x}{s^{a/b}} \\ &= \eta(x,s),\end{aligned} \tag{150}$$

and the similarity variable is then

$$\eta(x,s) = xs^{-\frac{a\log(\epsilon)}{\log(2)+2i\pi n}}. \tag{151}$$

Also,

$$\begin{aligned}\frac{w}{\tau^{c/b}} &= \frac{\epsilon^c \phi}{(\epsilon^b s)^{c/b}} \\ &= \frac{\phi}{s^{c/b}} \\ &= \nu(\eta),\end{aligned} \tag{152}$$

suggesting that we seek a solution of the RZSE with the form

$$\phi_s(x) = s^{\frac{c \log(\epsilon)}{\log(2)+2i\pi n}} \nu(\eta). \tag{153}$$

Since the RZSE is invariant under the transformation, it is to be expected that the solution will also be invariant under the variable transformation. Taking $a = c = \log^{-1}(\epsilon)$, the partial derivatives transform like

$$\begin{aligned}\frac{\partial}{\partial s}\phi_s(x) &= \frac{\partial}{\partial s}\Big(s^{\frac{1}{\log(2)+2i\pi n}}\Big)\nu(\eta) + \Big(s^{\frac{1}{\log(2)+2i\pi n}}\Big)\nu'(\eta)\frac{\partial \eta}{\partial s} \\ &= \frac{s^{-1+\frac{1}{\log(2)+2i\pi n}}}{\log(2)+2i\pi n}\Big[\nu(\eta) - \nu'(\eta)\Big],\end{aligned} \tag{154}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\phi_s(x) &= \Big(s^{\frac{1}{\log(2)+2i\pi n}}\Big)\nu'(\eta)\frac{\partial \eta}{\partial x} \\ &= \nu'(\eta),\end{aligned} \tag{155}$$

where

$$\frac{\partial \eta}{\partial s} = -\frac{s^{-1}}{2i\pi n + \log(2)}, \tag{156}$$

and

$$\frac{\partial \eta}{\partial x} = s^{-\frac{1}{2i\pi n+\log(2)}}. \tag{157}$$

The RZSE then reduces to the ODE

$$\Big[s^{-1} + \log(2) + 2i\pi n\Big]\nu(\eta) + \Big[-s^{-1} + 2\log(2)\eta + 4i\pi n\eta\Big]\nu'(\eta) = 0, \ \forall\, n \in \mathbb{Z}. \tag{158}$$

3.1 General Solution

The homogenous linear differential Eq. (158) is separable [54], viz.,

$$\frac{d\nu}{\nu} = \frac{2i\pi n + s^{-1} + \log(2)}{s^{-1} - 4i\pi n\eta - \eta\log(4)} d\eta. \tag{159}$$

Integrating on both sides, we obtain

$$\ln|\nu| = \quad c_1 - \frac{\Big(2i\pi n + s^{-1} + \log(2)\Big)\log\Big(s^{-1} - 4i\pi n\eta - \eta\log(4)\Big)}{4i\pi n + \log(4)}. \tag{160}$$

Exponentiating both sides,

$$|\nu| = \exp(c_1)\Big(s^{-1} - 4i\pi n\eta - \eta\log(4)\Big)^{-\frac{2i\pi n + s^{-1} + \log(2)}{4i\pi n + \log(4)}}. \tag{161}$$

Renaming the constant $\exp(c_1) = C$ and dropping the absolute value recovers the lost solution $\nu(\eta) = 0$, giving the general solution to Eq. (158)

$$\nu_n(\eta) = \quad C\Big(s^{-1} - 4i\pi n\eta - \eta\log(4)\Big)^{-\frac{2i\pi n + s^{-1} + \log(2)}{4i\pi n + \log(4)}}, \ \forall\, n \in \mathbb{Z},\ \forall\, C \in \mathbb{R}. \tag{162}$$

By setting $C = 1$, and using Eqs. (151) and (153) in Eq. (162), we obtain the general solution to the RZSE Eq. (96), written

$$\phi_s(x) = s^{\frac{1}{\log(2)+2i\pi n}} \left[\frac{1}{s} + s^{-\frac{1}{\log(2)+2i\pi n}}\Big(- x\log(4) - 4i\pi n x\Big)\right]^{-\frac{2\pi n s + is\log(2) + i}{4\pi n s - is\log(4)}}, \quad \forall\, n \in \mathbb{Z}. \tag{163}$$

4 Conclusion

In this study, we have discussed some modern developments in analytic number theory using quantum mechanical analogies. This was accomplished by developing a Schrödinger equation and analysing its dynamics in both configuration space and momentum space. A symmetrization procedure was implemented to study the eigenvalues of the system, and the expectation values were calculated with the analytic continuation of the eigenfunction. A Gelfand triplet, or rigged Hilbert space, was implemented to ensure that the eigenvalues are well defined. Moreover, a second quantization procedure was performed to obtain the equations of motion and an analytical expression for the eigenvalues. It was also demonstrated that the eigenvalues are holomorphic across the dense subspace of the Hilbert space. A normalized convergent expression for the analytic continuation of the eigenfunction was obtained, and a convergence test for the expression was performed using the second axiom of Kolmogorov. Within our framework, it was explicitly demonstrated that the eigenvalues of the eigenfunction are indeed real integers, and periodically zero in the complex plane. Finally, a general solution to Hamilton's equations were found by performing an invariant similarity transformation.

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