

BICOMBINANTS OF THE RATIONAL PLANE QUARTIC AND COMBINANT CURVES OF THE RATIONAL PLANE QUINTIC*

BY

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Introduction.

Each covariant locus of a rational plane curve contributes its share toward the covariant and invariant theory of rational curves of higher order by reason of the fact that osculants† of rational curves play the same rôle in regard to rational curves as polars do in regard to binary forms. Especially is this true of covariant points and covariant lines of the rational curve of order n (which we shall call R^n) because corresponding to each covariant line or point of an R^n is a series of covariant rational curves of any R^k , where $k > n$; the parametric equations of these covariant rational curves are easy to obtain. In the first section of this paper a new kind of covariant locus is defined for R^n and illustrated by important loci associated with the R^4 . Section II is devoted to combinant curves of the R^5 . The theory of combinant curves of the R^5 in the plane corresponds to the invariant theory of the R^5 in space; this correspondence is pointed out and illustrated. Finally a sufficient number of combinants and combinative m -ics of two line sections of the plane R^5 have been found to form the algebraic basis of an analytic treatment of combinant curves of the R^5 in the plane, which is shown to be identical with the invariant theory of the R^5 in space.

Section I. Bicombinants of the Rational Plane Quartic Curve.

Bicombinants.

Combinant curves of R^n have already been defined in these Transactions‡ as covariant loci of R^n which are combinants of two line sections of the R^n . Covariant loci of a slightly different kind are considered in this section, and as frequent reference to the above article is necessary, the letters used there to designate curves or expressions are employed with the same meaning in this

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† See THOMSEN, *American Journal of Mathematics*, vol. 32 (1910).

‡ *Important Covariant Curves and a Complete System of Invariants of the Rational Quartic Curve*, these Transactions, vol. 12 (1911), pp. 295–310.

present article. Also the results obtained there will be used here often without specific reference.

Let the R^4 be written parametrically

$$(1) \quad x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i \equiv f_i^4;$$

and the R^n

$$(2) \quad x_i = a_i t^n + nb_i t^{n-1} \dots \equiv f_i^n. \quad (i = 0, 1, 2).$$

The combinant curves of an R^n are expressible in terms of three-rowed determinants of the type

$$(3) \quad |abx| = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix}$$

by means of a translation scheme* already explained in detail and to be used also in the second section of this paper in considering a set of covariant curves of the R^5 . The combinants of the three binary forms f_i are expressible in terms of three-rowed determinants of the type

$$(4) \quad |abc| = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

There is a class of covariant curves of the R^n whose equations are composed of terms which are combinations of determinants of the types (3) and (4). Evidently these are not purely combinant curves as already defined, and inasmuch as such curves in form partake of the nature of the combinants of three binary forms as well as of the nature of combinant curves, they will be referred to as Bicombinant Curves, or Bicombinants, of the R^n . It may happen that a combinant curve and an invariant of the R^n together form a bicombinant; thus combinant curves are special cases of bicombinants. Such functions may be derived as follows: Let the parameters of a set of covariant points on an R^n be given by the binary form $(\gamma t)^n = 0$, whose coefficients are combinations of determinants of the type (4); also let a combinative m -ic of two line sections be $(\delta t)^m = 0$, whose coefficients are combinations of determinants of the type (3); each of these binary forms is unaltered by a linear transformation of the x 's; hence any simultaneous invariant* of $(\gamma t)^n = 0$ and $(\delta t)^m = 0$ is a covariant of the R^n and possesses the properties peculiar to bicombinants. We shall now apply this to the R^4 .

For convenience let

* Evidently any such function is unaltered by a linear transformation of the t 's.

$$(5) \quad \begin{cases} \alpha = |abx|, & \beta = |adx|, & \lambda = |acx|, & \mu = |bcx|, & \gamma = |aex|, \\ \alpha' = |dex|, & \beta' = |bex|, & \lambda' = |cex|, & \mu' = |cdx|, & \delta = |bdx|. \end{cases}$$

Among these ten Greek letters the following identities exist:

$$(6) \quad \begin{cases} \beta\mu - \lambda\delta + \alpha\mu' = 0, & \beta'\mu' - \lambda'\delta + \alpha'\mu = 0, \\ \alpha\lambda' - \beta'\lambda + \gamma\mu = 0, & \alpha'\lambda - \beta\lambda' + \gamma\mu' = 0, \\ \alpha\alpha' + \gamma\delta - \beta\beta' = 0. \end{cases}$$

The expression

$$(7) \quad (\beta - 3\mu)t^2 + (\gamma - 2\delta)t + (\beta' - 3\mu') = 0$$

is a combinative quadratic of two line sections of the R^4 . The osculant cubic of the R^4 at a point t' is an R^3 with three flexes on a line; if the equation of this flex line is formed and t' made variable, the result is (7). Hence for a given value of t , say t_1 , (7) becomes the equation of the flex line of the cubic osculant at t_1 ; for the coördinates of a point (7) becomes a quadratic whose roots are the names of the two cubic osculants whose flex lines pass through the given point. These flex lines of cubic osculants envelope a conic which is given parametrically by (7). The discriminant of (7) is the point equation of this conic which in our notation* is called $A - 12B = 0$. The conic on the flexes of the R^4 cuts out two other points which shall be called q_i . These are given by the quadratic

$$(8) \quad [2|acd| - |abc|]t^2 + [8|bcd| - |ace|]t + [2|bce| - |ade|] = 0;$$

for convenience let this be written

$$(9) \quad Q_0 t^2 + Q_1 t + Q_2 = 0.$$

The condition for (7) and (9) to be apolar is

$$(10) \quad L_1 = 2Q_0(\beta' - 3\mu') + 2Q_2(\beta - 3\mu) + (2\delta - \gamma)Q_1 = 0,$$

a covariant line of the R^4 , the locus of points such that tangents from any point of it to the conic $A - 12B = 0$ are apolar to the quadratic (8), i. e., L_1 cuts out (8) on the conic $A - 12B = 0$.

The R^4 has six flexes whose parameters are roots of the sextic

$$(11) \quad |abc|t^6 + 2|abd|t^5 + [3|acd| + |abc|]t^4 + [2|ace| + 4|bcd|]t^3 \\ + [3|bce| + |ade|]t^2 + 2|bde|t + |cde| = 0.$$

* L. c., p. 1.

A combinative sextic of two line sections is

$$(12) \quad \alpha t^6 + 3\lambda t^5 + 3[\beta + 2\mu]t^4 + [\gamma + 8\delta]t^3 + 3[\beta' + 2\mu']t^2 + 3\lambda't + \alpha' = 0:$$

for a given t , say t_1 , (12) becomes the equation of the tangent to the R^4 at that point; for the coördinates of a point it yields six roots which are the parameters of the six tangents from the given point to the R^4 . The apolarity condition of (11) and (12) yields a second covariant line which if reduced by identities of the type

$$(13) \quad |abc| \alpha' - |abd| \lambda' = |bcd| \gamma - |acd| \beta'$$

becomes the same as L_1 . Hence, *the locus of points such that tangents drawn from it to the R^4 are apolar to the flexes is the line L_1 , which cuts out the q_i on $A - 12B = 0$.*

Another set of covariants which can be expressed as bicombinants is illustrated by the following example. The two tangents to $A - 12B = 0$ at the two points (9) may be found by substituting t_1 and t_2 in (7) for t , multiplying the resulting expressions, and replacing the symmetric functions of t_1 and t_2 by the proper coefficients of (9). Calling the product of these tangents Q' (evidently they are the flex lines of the cubic osculants of the points q_i) we find

$$(14) \quad \begin{aligned} Q' = & [\beta^2 - 6\beta\mu + 9\mu^2]Q_2^2 + [\beta'^2 - 6\beta'\mu' + 9\mu'^2]Q_0^2 + [\gamma^2 - 4\gamma\delta + 4\delta^2]Q_0Q_2 \\ & - [\beta\gamma - 2\beta\delta - 3\mu\gamma + 6\mu\delta]Q_1Q_2 - [\beta'\gamma - 2\beta'\delta - 3\mu'\gamma - 6\mu'\delta]Q_0Q_1 \\ & + [\beta\beta' - 3\beta\mu' - 3\beta'\mu + 9\mu\mu'] [Q_1^2 - 2Q_0Q_2] = 0. \end{aligned}$$

The following relation exists:

$$(15) \quad L_1^2 - 4Q' = [Q_1^2 - 4Q_0Q_2][A - 12B];$$

this is simply the conic $A - 12B = 0$ expressed in terms of two tangents and the chord of contact; $Q_1^2 - 4Q_0Q_2 = 0$ is the condition for the two points q_i to unite. If equation (12) had been used instead of (9), by the same process the equation of the two tangents to the R^4 at the points q_i could have been obtained; in this manner many important covariants of R^n can be expressed in bicombinant form.

The apolarity condition of (9) cubed and (12) gives a covariant line L_3 , the locus of points such that tangents from it to the R^4 are apolar to the points q_i each taken three times. It is evident from geometrical considerations that the two q_i are on this line when $Q_1^2 - 4Q_0Q_2 = 0$. It will be shown that the covariant line of the R^4 which is on the q_i , and two other points which we shall call r_i , is a member of the pencil

$$(16) \quad KL_1 + L_3 = 0,$$

where L_3 calculated from (9) and (12) is

$$(17) \quad \begin{aligned} L_3 = & 20[Q_0^3\alpha' + Q_2^3\alpha] - 30[Q_0^2Q_1\lambda' + Q_2^2Q_1\lambda] \\ & + 12[Q_0Q_1^2 + Q_0^2Q_2][\beta' + 2\mu'] + 12[Q_2Q_1^2 + Q_0Q_2^2][\beta + 2\mu] \\ & - Q_1[Q_1^2 + 6Q_0Q_2][\gamma + 8\delta] = 0. \end{aligned}$$

The methods used in finding the equation of the line determined by the q_i are the same as those used to prove the two following theorems in regard to the R^4 in a new way. Two combinant curves of the R^4 are:

$$(18) \quad A = 12\lambda\lambda' - 48\mu\mu' + \gamma^2 + 16\delta^2 - 8\alpha\alpha' - 8\beta\beta' = 0,$$

which is the conic touching the six flex tangents of the R^4 ; and

$$(19) \quad B = \lambda\lambda' - \mu\mu' - \beta\mu' - \beta'\mu + \delta^2 - \alpha\alpha' = 0,$$

which is the equation of the conic described by the intersections of the flex tangents of cubic osculants of the R^4 . The parameters of the six flexes of the R^4 are given by equation (11), a binary sextic which is a simultaneous covariant of the two binary quartics* which symmetrically represent the R^4 , or which form a pencil of binary quartics—the fundamental involution of the R^4 . Every covariant† of a binary form, or every simultaneous covariant of two binary forms can be obtained when its seminvariant, or leading coefficient, is given. Equation (11) is really the jacobian of the two quartics just referred to. Any member of the pencil of conics

$$(20) \quad A + K'B = 0$$

cuts out a set of covariant parameters on the R^4 , and each octavic so obtained is a covariant of the two quartics of the fundamental involution of the R^4 . Hence, *if any member of the pencil (20) can be found which cuts the R^4 in eight parameters given by a binary octavic whose leading coefficient is divisible by $|abc|$, the leading coefficient of (11), then equation (11) must be a factor of this octavic and the flexes of the R^4 must be on the corresponding conic of (20).* Evidently any term in A or B involving

$$(21) \quad a, \beta, \gamma, \text{ or } \lambda$$

cannot contribute a term towards the coefficient of t^8 of the binary octavic. Neglecting the terms of A and B which involve the letters (21), we have

* W. F. MEYER, *Apolarität und rationale Curven*, p. 9.

† SALMON, *Higher Algebra*, Fourth Edition, Lesson XIII, 13.

$$(22) \quad A = -48\mu\mu' + 16\delta^2,$$

$$(23) \quad B = -\mu\mu' - \beta'\mu + \delta^2.$$

The combination

$$(24) \quad A - 16B = 0$$

is a member of the pencil (20) cutting out of R^4 eight points given by a binary octavic whose leading coefficient* is divisible by $|abc|$. Hence (24) is the conic on the flexes of the R^4 . Incidentally, this amounts to a proof of the fact that they do lie on a conic. Also, *if without imposing a condition on the curve we find that one of a covariant set of points lies on a particular covariant curve, all of this set lie on the curve.* If the parameters 0 and ∞ are assigned to the points of contact of a double tangent, the equations of the R^4 may be made to assume the form

$$(25) \quad \begin{cases} x_0 = a_0t^4 + 4b_0t^3 + 6c_0t^2 + 4d_0t, \\ x_1 = 4b_1t^3 + 6c_1t^2 + 4d_1t + e_1, \\ x_2 = 6c_2t^2. \end{cases}$$

Calculating the values of the determinants (5) for the point (0 1 0), we find

$$(26) \quad A = -48\mu\mu', \quad B = -\mu\mu'.$$

The conic

$$(27) \quad A - 48B = 0$$

is on one point of contact of a double tangent and must be on all such contacts.

These two proofs illustrate the methods to be used in finding the equation of the covariant line on the q_i and the process employed in the second section in finding the equation of the combinant cubic on the flexes of the R^5 .

Applying what we have just stated to the pencil of lines (16), we find that L_4 , the covariant line of the R^4 on the q_i , is defined as follows:

$$(28) \quad L_3 + 4[Q_1^2 - 4Q_0Q_2]L_1 = 5L_4.$$

Hence,

$$(29) \quad \begin{aligned} L_4 = & 4[Q_0^3\alpha' + Q_2^3\alpha] - 6[Q_0^2Q_1\lambda' + Q_2^2Q_1\lambda] + [4Q_0Q_1^2 - 4Q_0^2Q_2]\beta' \\ & + [4Q_2Q_1^2 - 4Q_0Q_2^2]\beta + 24Q_0^2Q_2\mu' + 24Q_0Q_2^2\mu - 16Q_0Q_1Q_2\delta \\ & + [2Q_0Q_1Q_2 - Q_1^3]\gamma = 0. \end{aligned}$$

If the parametric equations of the R^4 are substituted in (29), and (9) factored out, we find the two r_i are given by the quadratic

* By the same reasoning it may be shown that the two q_i are on the conic (24).

$$\begin{aligned}
 (30) \quad & [4Q_0^2|ade| + 6Q_0Q_1|ace| + 4(Q_1^2 - Q_0Q_2)|abe| + 24Q_2^2|abc| + 24Q_0Q_2|acd| \\
 & - 16Q_1Q_2|abd|]t^2 + [16Q_0^2|bde| + 16Q_2^2|abd| - 24Q_0Q_1|bce| \\
 & - 24Q_1Q_2|acd| - 4Q_1Q_2|abe| - 4Q_0Q_1|ade| + 96Q_0Q_2|bcd| \\
 & + 6Q_1|ace|]t + [4Q_2^2|abe| - 6Q_2Q_1|ace| + 4(Q_1^2 - Q_0Q_2)|ade| \\
 & + 24Q_0^2|cde| + 24Q_2Q_0|bce| - 16Q_0Q_1|bde|] = 0.
 \end{aligned}$$

The condition for (9) and (30) to be harmonic is C'_1 , i. e., the condition for four collinear flexes. Hence, when the conic $A - 16B = 0$ degenerates,* four flexes are on one line and the two remaining flexes and the two q_i on the other, *the latter two flexes and q_i being harmonic*, since a simple substitution shows that in this case the r_i coincide with the two flexes on the line with the q_i .

A Covariant Point of the R^4 .

The intersection of L_1 and L_3 is a covariant point P of the R^4 , which possesses the following properties: (1) the six tangents from P to R^4 have their parameters apolar to the flex parameters and to the parameters of the q_i each taken three times; (2) the line cutting out the two q 's on R^4 passes through P ; (3) the line cutting out the two q_i on the conic $A - 12B = 0$ passes through P .

The parameters of the two points on $A - 12B = 0$ whose tangents to $A - 12B = 0$ meet at P can be found by solving (10) and (17) for x_0 , x_1 , and x_2 and substituting these values in equations (7). The result is a quadratic in t whose roots are the required parameters. The coördinates of P are of degree 16 in the coefficients of the parametric equations of the R^4 ; hence the quadratic just mentioned has coefficients of degree six in the three-rowed determinants of the type $|abc|$. Since this quadratic is apolar to (8), the line cutting out the parameters which are its roots on $A - 12B = 0$ passes through the intersection of the tangents to $A - 12B = 0$ at the two points q_i . If we recall that the q_i and the flexes of the R^4 lie on a conic ($A - 16B = 0$), and that the flex parameters are given by the jacobian of two binary quartics whose third transvectant yields the parameters of the q_i , the properties found for P become more interesting because they suggest the possibility of constructing six points which are related to six other points and to two associated points in the same way that the six points of contact of tangents from P to R^4 are related to the six flexes and the two q_i .

* See A. BRILL, *Mathematische Annalen*, vol. 17 (1880). In this article BRILL shows that if an R^4 has three flexes on a line, it must have four on the line.

Bicombinant Conics.

Adjoint curves of R^n are curves which pass through the nodes and certain other points necessary to account for their complete intersection with the R^n . For instance the adjoint conic of the R^4 cuts in six nodal parameters and in two other points, say S_1 and T_1 ; if S_1 and T_1 are definite points then this curve is known as the adjoint conic of the R^4 through the points S_1 and T_1 , or simply as the adjoint of S_1 and T_1 ; if the conic cuts out the nodes and touches the R^4 at S_1 , the conic is called the adjoint of S_1 . An adjoint cubic of the R^4 cuts out the nodal parameters and six other points. Hence for every pair of points on the R^4 there is a definite adjoint conic; for every set of six points there is a definite adjoint cubic. In general such curves are not covariants of the R^4 , but if the points other than the nodes which are cut out of the R^4 by an adjoint conic or cubic happen to be a set of covariant parameters on the R^4 , this adjoint is certainly a covariant curve of the R^4 as it cuts the R^4 in only covariant points. Stahl* has indicated how the adjoint curves of orders $n-1$ and $n-2$ of an R^n can be written down as determinants of order $3n-2$, although he says nothing of covariant adjoints. There is a definite conic on the nodes and the points q_i ; a second conic on the nodes and the points r_i , and a definite conic on the nodes and any covariant set of points given by a quadratic. The adjoint of the q_i will be written out in bicombinant form, and any other may be obtained by substituting instead of Q_0 , Q_1 , and Q_2 the corresponding coefficients of any other quadratic whose roots are a pair of covariant parameters.

The bicombinant conic on the nodes and the points q_i of the R^4 of equation (1) is

$$\begin{aligned}
 & Q_0 \left\{ \begin{aligned} & 24 | abc | (36\lambda'^2 - 96\alpha'\mu' - 16\alpha'\beta') \\ & \quad - 16 | abd | (24\beta'\lambda' - 64\alpha'\delta - 4\alpha'\gamma) \\ & + 4 | abe | (16\beta'^2 - 96\alpha'\mu - 6\lambda'\gamma) + 24 | acd | (6\lambda'\gamma - 16\alpha'\beta) \\ & \quad - 6 | ace | (4\beta'\gamma - 24\alpha'\lambda) + 4 | ade | (\gamma^2 - 16\alpha\alpha') \end{aligned} \right\} \\
 (32) + & Q_1 \left\{ \begin{aligned} & 96 (| abc | \alpha'\gamma + | cde | \alpha\gamma) - 256 (| abd | \alpha'\beta + | bde | \alpha\beta') \\ & - 16 (| abe | \beta'\gamma + | ade | \beta\gamma) + 96 (| abe | \beta\lambda' + | ade | \beta'\lambda) \\ & + 576 (| acd | \alpha'\lambda + | bce | \alpha\lambda') + 6 | ace | (\gamma^2 - 36\lambda\lambda') \end{aligned} \right\} \\
 + & Q_2 \left\{ \begin{aligned} & 24 | cde | (36\lambda^2 - 96\alpha\mu - 16\alpha\beta) - 16 | bde | (24\beta\lambda - 64\alpha\delta - 4\alpha\gamma) \\ & + 4 | ade | (16\beta^2 - 96\alpha\mu' - 6\lambda\gamma) + 24 | bce | (6\lambda\gamma - 16\alpha\beta') \\ & - 6 | ace | (4\beta\gamma - 24\alpha\lambda') + 4 | abe | (\gamma^2 - 16\alpha\alpha') \end{aligned} \right\} = 0.
 \end{aligned}$$

* *Mathematische Annalen*, vol. 38 (1891), pp. 583-4.

Bicombinant Cubics.

Corresponding to each set of six covariant parameters on the R^4 is a definite covariant cubic cutting out these and the nodes from R^4 . Hence the cubic on the nodes and flexes of the R^4 can be found in this way. Also the adjoint cubic of the nodes themselves is simply the equation of the three lines forming the natural reference triangle of the R^4 . This cubic cuts out of the R^4 the nodal sextic counted twice. We shall write out the equation of the cubic on the nodes and flexes of the R^4 , understanding that the six flex parameters are given by

$$(33) \quad F_0 t^6 + F_1 t^5 + F_2 t^4 + F_3 t^3 + F_4 t^2 + F_5 t + F_6 = 0.$$

Its equation is

$$(34) \quad \left\{ \begin{array}{l} F_0 \left(4\alpha' \gamma^2 + 64\alpha' \gamma \delta - 48\beta' \lambda' \gamma - 768\alpha' \beta' \mu + 64\beta'^3 + 144\gamma \lambda' \mu' + 864\lambda'^2 \mu \right) \\ - 384\beta' \lambda' \delta + 1024\alpha' \delta^2 - 64\alpha\alpha'^2 - 384\alpha' \mu' \beta + 144\alpha' \lambda \lambda' - 2304\alpha' \mu \mu' \\ + F_1 \left(96\alpha' \beta' + 576\alpha' \mu' \lambda - 216\lambda \lambda'^2 - 16\alpha' \beta \gamma - 256\alpha' \beta \delta + 96\beta \beta' \lambda \right. \\ \left. + 6\gamma^2 \lambda' + 96\alpha' \gamma \mu - 16\beta'^2 \gamma \right) \\ + F_2 \left(- 64\alpha\alpha' \beta' - 384\alpha\alpha' \mu' + 144\alpha\lambda'^2 + 64\alpha' \beta^2 - 24\beta \gamma \lambda' \right. \\ \left. - 24\alpha' \gamma \lambda + 4\beta' \gamma^2 \right) \\ + F_3 \left((32\alpha\alpha' \gamma + 256\alpha\alpha' \delta - 96\alpha\beta' \lambda' - 96\alpha' \beta \lambda + 36\lambda \lambda' \gamma - \gamma^3) \right. \\ \left. + F_4^* (\quad) + F_5 (\quad) + F_6 (\quad) = 0; \right\} \end{array} \right.$$

evidently the covariant combinant cubic on the nodes and any set of six covariant parameters can be obtained by substituting the coefficients of the sextic yielding these covariant parameters for the F 's in (34). Equation (30) squared multiplied by (9) gives a sextic; if the coefficients of this sextic be substituted in (34) for the F 's, the result is a covariant bicombinant cubic cutting out the nodes, q_i , and touching the R^4 at the points r_i ; the cube of equation (9) substituted for (33) makes (34) a covariant bicombinant cubic on the nodes, osculating the R^4 at each point q .

Section II. Combinant Curves of the Rational Plane Quintic and Their Relation to Invariants of the Rational Quintic Curve in Space.

Let the R^5 be written parametrically

$$(1) \quad x_i = a_i t^5 + 5b_i t^4 + 10c_i t^3 + 10d_i t^2 + 5e_i t + f_i \quad (i = 0, 1, 2).$$

* Coefficients of F_4 , F_5 , and F_6 are obtained by exchanging primed and unprimed letters in F_2 , F_1 , and F_0 .

Cutting (1) by two lines

$$(2) \quad (\xi x) = 0 \quad \text{and} \quad (\eta x) = 0$$

we obtain the two binary quintics

$$(3) \quad (\alpha t)^5 \equiv (a\xi) t^5 + 5 (b\xi) t^4 + 10 (c\xi) t^3 \dots = 0,$$

$$(4) \quad (\beta t)^5 \equiv (a\eta) t^5 + 5 (b\eta) t^4 \dots = 0.$$

The combinants of (3) and (4) are expressible in terms of two-rowed determinants of the matrix

$$(5) \quad \begin{vmatrix} (a\xi) & (b\xi) & (c\xi) & (d\xi) & (e\xi) & (f\xi) \\ (a\eta) & (b\eta) & (c\eta) & (d\eta) & (e\eta) & (f\eta) \end{vmatrix}.$$

If x_0 , x_1 , and x_2 are substituted for the coördinates of the point in which the lines (2) intersect, any two-rowed determinant of (5) becomes a three-rowed determinant; for instance,

$$(6) \quad \begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} \quad \text{becomes} \quad \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} \equiv |abx|$$

by this process. Hence following out this scheme of interpretation, we have Covariant Curves of the R^5 corresponding to combinants of (3) and (4).

The following abbreviations will be found convenient:

$$(7) \quad \begin{cases} \alpha = |abx|, & \beta = |acx|, & \gamma = |adx|, & \delta = |aex|, & \lambda = |bcx|, \\ \alpha' = |efx|, & \beta' = |dfx|, & \gamma' = |cfx|, & \delta' = |bfx|, & \lambda' = |dex|, \\ \mu = |bdx|, & \mu' = |cex|, & \psi = |afx|, & \varphi = |bex|, & \chi = |cdx|. \end{cases}$$

Among these Greek letters exist the identities

$$(8) \quad \begin{cases} \alpha\alpha' - \delta\delta' + \psi\varphi = 0, & \beta\beta' - \gamma\gamma' + \psi\chi = 0, & \gamma\gamma' - \mu\mu' + \varphi\chi = 0, \\ \alpha\chi - \beta\mu + \gamma\lambda = 0, & \alpha'\chi' - \beta'\mu' + \gamma'\lambda' = 0, \\ \alpha\mu' - \beta\varphi + \delta\lambda = 0, & \alpha'\mu - \beta'\varphi + \delta'\lambda' = 0, \\ \alpha\gamma' - \beta\delta' + \psi\lambda = 0, & \alpha'\gamma - \beta'\delta + \psi\lambda' = 0, \\ \alpha\lambda' - \gamma\varphi + \mu\delta = 0, & \alpha'\lambda - \gamma'\varphi + \mu'\delta' = 0, \\ \alpha\beta' - \gamma\delta' + \psi\mu = 0, & \alpha'\beta - \gamma'\delta + \psi\mu' = 0, \\ \beta\lambda' - \gamma\mu' + \delta\chi = 0, & \beta'\lambda - \gamma'\mu + \delta'\chi = 0. \end{cases}$$

The first transvectant of (3) and (4) or their jacobian is

$$(9) \quad \alpha t^8 + 4\beta t^7 + (6\gamma + 10\lambda)t^6 + (4\delta + 20\mu)t^5 + \psi + 15\varphi + 20\chi)t^4 \\ + (4\delta' + 20\mu')t^3 + (6\gamma' + 10\lambda')t^2 + 4\beta't + \alpha' = 0;$$

for a given t this becomes the equation of the tangent to the R^5 at that point; for the coördinates of a point it yields the parameters of the eight tangents to the R^5 from the given point.

The third transvectant of (3) and (4) is

$$(10) \quad (\gamma - 3\lambda)t^4 + (2\delta - 4\mu)t^3 + (\psi + \varphi - 8\chi)t^2 \\ + (2\delta' - 4\mu')t + (\gamma' - 3\lambda') = 0;$$

the osculant cubic of the R^5 has three flexes on a line, and if t_1 is substituted in (10) for t the result is the flex line of the cubic osculant at the point t_1 ; for the coördinates of a point (10) yields the parameters of four points, the flex lines of whose cubic osculants are on the given point. These flex lines of cubic osculants envelope a rational class curve of order four which shall be referred to as the ρ^4 of the R^5 . From (10) the parametric equations of this ρ^4 are

$$(11) \quad \xi_i = [(a_u d_v) - 3(b_u c_v)]t^4 + [2(a_u e_v) - 4(b_u d_v)]t^3 \\ + [(a_u f_v) + (b_u e_v) - 8(c_u d_v)]t^2 + [2(b_u f_v) - 4(c_u e_v)]t \\ + [(c_u f_v) - 3(d_u e_v)] \quad (i, u, v = 0, 1, 2).$$

This curve is of the fourth class and sixth order and is therefore a rational sextic with six cusps.

The fifth transvectant of (3) and (4) is

$$(12) \quad P = \psi - 5\varphi + 10\chi = 0 = 5P';$$

this is the well-known covariant line of the R^5 which cuts out a set of the fundamental involution.

Twelve times the g_2 of (10) we call Q ; hence

$$(13) \quad Q = 12\gamma\gamma' - 36(\gamma\lambda')^* + 108\lambda\lambda' - 12\delta\delta' + 24(\mu\delta') - 48\mu\mu' \\ + (\psi + \varphi - 8\chi)^2 = 0;$$

this is the conic on the cusps of the ρ^4 .

One hundred and forty times the g_2 of (9) we call T , and we find that

$$(14) \quad 3T = 28P^2 - 25Q;$$

evidently any two members of this pencil of conics may be so combined as to form the covariant line squared, hence *this pencil of conics has double contact and the covariant line P is on the two points of contact.*

* Such an expression as $\gamma\lambda'$ when in parenthesis carries with it the conjugate expression $\gamma'\lambda$.

Two hundred and sixteen times the g_3 of (10) we call D_3 , which written in full is

$$(15) \quad \begin{aligned} D_3 = & (\psi + \varphi - 8\chi)[36\gamma\gamma' - 108(\gamma\lambda') + 324\lambda\lambda' + 18\delta\delta' \\ & - 36(\mu\delta') + 72\mu\mu'] - 54(\gamma\delta'^2) + 216(\mu'\delta'\gamma) - 216(\mu'^2\gamma) \\ & + 162(\lambda\delta'^2) - 648(\mu'\delta'\lambda) + 648(\lambda\mu'^2) - (\psi + \varphi - 8\chi)^3 = 0. \end{aligned}$$

Calculating an invariant of the third order in the coefficients for a binary octavic written in the usual way with binomial coefficients, I find

$$(16) \quad \begin{aligned} I_3 = & \alpha_0\alpha_4\alpha_8 - 4(\alpha_0\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_8) + 3(\alpha_0\alpha_6^2 + \alpha_2^2\alpha_8) \\ & - 8(a_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_7) + 12\alpha_1\alpha_4\alpha_7 - 22\alpha_2\alpha_4\alpha_6 \\ & + 24(\alpha_2\alpha_5^2 + \alpha_3^2\alpha_6) - 36\alpha_3\alpha_4\alpha_5 + 15\alpha_4^3 = 0. \end{aligned}$$

Every binary octavic has a covariant octavic of degree two in its coefficients, which yields eight points whose fourth polars as to the octavic are self-apolar sets. The apolarity condition of this covariant and the original octavic is a multiple of the above invariant. Applying what has been stated to (9), we find that if $14^3 \cdot 25I_3$ is called $3H$,

$$(17) \quad \begin{aligned} 3H = & (5\psi + 75\varphi + 100\chi)[196\alpha\alpha' + 588\beta\beta' - 198\gamma\gamma' - 330(\gamma\lambda') \\ & - 550\lambda\lambda' - 36\delta\delta' - 180(\mu\delta') - 900\mu\mu'] - 9,800(\alpha\beta'\delta') \\ & - 49,000(\alpha\beta'\mu') + 9,450(\alpha\gamma'^2) + 31,500(\alpha\gamma'\lambda') \\ & + 26,250(\alpha\lambda'^2) - 4,200(\beta\delta'\gamma') - 7,000(\beta\lambda'\delta') \\ & - 21,000(\beta\mu'\gamma') - 35,000(\beta\mu'\lambda') + 1,800(\gamma\delta'^2) \\ & + 18,000(\gamma\mu'\delta') + 45,000(\gamma\mu'^2) + 3,000(\lambda\delta'^2) \\ & + 30,000(\mu'\delta'\lambda) + 75,000(\lambda\mu'^2) + 3(\psi + 15\varphi + 20\chi)^3 = 0. \end{aligned}$$

The quartic osculant of the R^5 at a point t' is

$$(18) \quad \begin{aligned} x_i = & (a_i t' + b_i) t^4 + 4(b_i t' + c_i) t^3 + 6(c_i t' + d_i) t^2 \\ & + 4(d_i t' + e_i) t + (e_i t' + f_i) \quad (i = 0, 1, 2). \end{aligned}$$

This is an R^4 , and if the conic A as described in Section I is formed for (18), the result will involve t' to the fourth power; making t' variable we obtain

$$(19) \left\{ \begin{aligned} & [12\beta\mu' - 48\lambda\chi + \delta^2 + 16\mu^2 - 8\gamma\varphi - 8\alpha\lambda']t^4 + [12\beta\gamma' + 4\beta\lambda' \\ & \quad + 4\gamma\mu' - 36\lambda\mu' - 16\mu\chi + 24\mu\varphi + 2\delta\psi + 6\delta\varphi - 8\alpha\beta' - 8\gamma\delta']t^3 \\ & \quad + [4\beta\beta' + 4\gamma\gamma' - 44\lambda\lambda' + 12(\gamma\lambda') - 32\chi^2 - 24\mu\mu' - 6\delta\delta' \\ & \quad + \psi^2 + 2\psi\varphi + 9\varphi^2 + 32\varphi\chi - 8\alpha\alpha' - 8(\mu\delta')]t^2 + [12\beta'\gamma \\ & \quad + 4\beta'\lambda + 4\gamma'\mu - 36\lambda'\mu - 16\mu'\chi + 24\mu'\varphi + 2\delta'\psi - 6\delta'\varphi \\ & \quad - 8\alpha'\beta - 8\gamma'\delta]t + [12\beta'\mu - 48\lambda'\chi + \delta'^2 + 16\mu'^2 - 8\gamma'\varphi \\ & \quad \quad \quad - 8\alpha'\lambda] = 0; \end{aligned} \right.$$

for a given t , (19) becomes the equation of the conic on the flex tangents of the osculant quartic at that point; for the coördinates of a point, (19) becomes a quartic whose roots are the parameters of four osculant quartics whose conics A are on the given point. Likewise, by using the conic B we obtain from (18) by the same process

$$(20) \left\{ \begin{aligned} & [\beta\mu' - \lambda\chi - \gamma\chi - \varphi\lambda + \mu^2 - \alpha\lambda']t^4 + [\beta\gamma' - \mu'\lambda + \varphi\mu - \delta\chi \\ & \quad - \lambda\delta' - \alpha\beta']t^3 + [\gamma\gamma' - \alpha\alpha' - \lambda\lambda' - \mu\mu' - (\mu\delta') + \varphi^2]t^2 \\ & \quad + [\beta'\gamma - \mu\lambda' + \varphi\mu' - \delta'\chi - \lambda'\delta - \alpha'\beta]t + [\beta'\mu - \lambda'\chi \\ & \quad \quad \quad - \gamma'\chi - \varphi\lambda' + \mu'^2 - \alpha'\lambda] = 0; \end{aligned} \right.$$

for a given t this becomes the conic B for the osculant quartic at that point; for the coördinates of a point, (20) becomes a quartic whose roots are the parameters of four osculant quartics whose conics B are on the given point. Both (19) and (20) are combinative quartics of two line sections of the R^5 . Six times the apolarity condition of (10) and (19) we call A_3 , and six times the apolarity condition of (10) and (20) we call B_3 . These combinant cubics are expressible in terms of P , Q , D_3 , and H , which are not connected by any linear relation. We find that

$$(21) \quad 46,250B_3 = 1,375D_3 - 81H + 4,250PQ - 2,744P^3,$$

and

$$(22) \quad 7,875A_3 = 125D_3 - 162H + 13,650PQ - 5,488P^3.$$

It is much easier to work with A_3 and B_3 than with D_3 and H , hence other combinant cubics will be expressed in terms of A_3 , B_3 , P , and Q .

Every binary octavic has four points whose second polars as to itself form self-apolar sets. These points are given by a quartic covariant of degree two in the coefficients of the octavic. Forming this covariant for (9) and requiring it to be apolar to (10), the resulting condition is proportional to

$$(23) \quad 27PQ - 170B_3 - 30A_3 = 0.$$

If equation (9) is required to be apolar to the square of (10), the condition may be expressed as

$$(24) \quad 130B_3 - 5A_3 + 8PQ = 0.$$

Of course, we know from the theory of the binary quartic that the condition for (10) and its Hessian to be apolar is a multiple of D_3 . Also, applying to any member of the pencil of conics $A + KB = 0$ the same process as was applied to A would yield a combinative quartic of two line sections whose coefficients are of the same degree as (19); requiring any one of these to be apolar to (10) would yield a combinative cubic expressible in terms of A_3 and B_3 .

Cubic on the Flexes of the R^4 .

In general no relation of the form

$$(27) \quad aH + bD_3 + cP^3 + dPQ = 0$$

exists, as may easily be shown by making $b_i = e_i = 0$ in (1) and comparing the resulting values of the letters used in (27). But if we can find a cubic of the system (27) that cuts out *one flex without imposing any condition on the curve, this cubic cuts out all the flexes*. We find only one system of values of a , b , c , and d which satisfy these conditions; hence there is only one *combinant* cubic on the flexes of the R^5 . Its equation may be written in either of the following forms,

$$(28) \quad 81H - 3,125D_3 + 2,296P^3 - 2,100PQ = 0,$$

or

$$(29) \quad 375A_3 - 7,875B_3 - 100PQ - 32P^3 = 0.$$

The Flex Tangents of the R^5 .

The product of the nine flex tangents of the R^5 can be obtained by finding the condition that there shall be a member of the pencil

$$(30) \quad (\alpha t)^5 + k(\beta t)^5 = 0$$

which contains a cubed factor. This condition is imposed by requiring the vanishing of the matrix

$$(31) \quad (\alpha t)^3 (\beta t)^3 \begin{vmatrix} \alpha_1^2 & \alpha_1 \alpha_2 & \alpha_2^2 \\ \beta_1^2 & \beta_1 \beta_2 & \beta_2^2 \end{vmatrix};$$

that is, a value of t must simultaneously satisfy the three sextics

$$(32) \quad \begin{aligned} (\alpha t)^3 (\beta t)^3 \begin{vmatrix} \alpha_1^2 & \alpha_1 \alpha_2 \\ \beta_1^2 & \beta_1 \beta_2 \end{vmatrix} &= 0, & (\alpha t)^3 (\beta t)^3 \begin{vmatrix} \alpha_1^2 & \alpha_2^2 \\ \beta_1^2 & \beta_2^2 \end{vmatrix} &= 0, \\ (\alpha t)^3 (\beta t)^3 \begin{vmatrix} \alpha_1 \alpha_2 & \alpha_2^2 \\ \beta_1 \beta_2 & \beta_2^2 \end{vmatrix} &= 0. \end{aligned}$$

If it is understood that the translation scheme explained in the first part of this section has been carried out, these sextics of (32) can be written in the form

$$(33) \quad \begin{cases} \alpha t^6 + 3\beta t^5 + (3\gamma + 6\lambda)t^4 + (\delta + 8\mu)t^3 + (3\varphi + 6\chi)t^2 + 3\mu't + \lambda' = 0, \\ \beta t^6 + (3\gamma + 3\lambda)t^5 + (3\delta + 9\mu)t^4 + (\psi + 9\varphi + 8\chi)t^3 \\ \quad + (3\delta' + 9\mu')t^2 + (3\gamma' + 3\lambda')t + \beta' = 0, \\ \lambda t^6 + 3\mu t^5 + (3\varphi + 6\chi)t^4 + (\delta' + 8\mu')t^3 + (3\gamma' + 6\lambda')t^2 + 3\beta't + \alpha' = 0. \end{cases}$$

If equations (33) are multiplied by t , and then by t^2 , we obtain 6 equations which together with any two of (33) may be solved for $t^8, t^7, t^6, t^5, t^4, t^3, t^2$, and t , and these values substituted in the third equation of (33). Evidently this is the same as the 9-rowed determinant of the 9 simultaneous equations in eight variables obtained in the above way. The method used here for finding the equation of the 9 flex tangents of the R^5 can be used for finding the product of the $3n - 6$ flex tangents of the R^n , the general application being so obvious that a formal statement is unnecessary.

The point equation of the R^5 is obtained as the eliminant of (3) and (4). As Salmon gives this explicitly as a five-rowed determinant,* it need not be given here. The work of finding combinants of (3) and (4) could be extended indefinitely, but we believe that those already written out together with invariants and simultaneous invariants† of equations (9), (10), (19) and (20) are a sufficient algebraic basis for the analytic treatment of covariant combinant loci of the R^5 in the plane, which is the same as the invariant theory of the R^5 in space. Hence we shall point out this relation between the combinant curves of the R^5 in the plane and the invariants of the R^5 in space and illustrate the principle by means of several examples.

* SALMON, *Higher Algebra*, Fourth Edition, p. 82.

† Extensive use can be made of the system of combinants of two binary quartics found in SALMON, *Higher Algebra*, Fourth Edition, pp. 219-223.

Invariants of the R^5 in Space.

It is shown in MEYER's *Apolarität und rationale Curven* * that every R^n in d dimensions is symmetrically represented by $n - d$ binary forms of order n ; when $n - d = 2$ this means that two binary forms of degree n symmetrically represent a R^n in $n - 2$ dimensions; that is, two binary quartics symmetrically represent an R^4 in the plane; two binary quintics symmetrically represent the R^5 in space. In the plane all line sections of the R^4 are apolar to the pencil of quartics which symmetrically represent the curve; in space any linear combination of the two binary quintics which symmetrically represent the R^5 in space is apolar to all plane sections of this R^5 . By the application of Grassmann's theorem it is possible by a simple substitution to obtain the combinants of the two binary quintics associated with the R^5 in space from two other binary quintics; taking the latter two binary quintics as (3) and (4), the combinants of the two former quintics which are invariants of the R^5 in space are obtained in the following manner. Let the two binary quintics (3) and (4) be rewritten in the form

$$(34) \quad \begin{aligned} at^5 + 5bt^4 + 10ct^3 + 10dt^2 + 5et + f &= 0, \\ a't^5 + 5b't^4 + 10c't^3 + 10d't^2 + 5e't + f' &= 0; \end{aligned}$$

let the parametric equations of the R^5 in space be written *without binomial coefficients* in the form

$$(35) \quad x_i = a_i t^5 + b_i t^4 + c_i t^3 + d_i t^2 + e_i t + f_i \quad (i = 0, 1, 2, 3).$$

The matrix of coefficients of (34) is

$$(36) \quad \begin{vmatrix} a & b & c & d & e & f \\ a' & b' & c' & d' & e' & f' \end{vmatrix};$$

the matrix of coefficients of (35) is

$$(37) \quad \begin{vmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{vmatrix}$$

If for any two-rowed determinant of (36) the four-rowed determinant of (37) (obtained by striking out the two columns of (37) which contain the

* See in particular § 3.

letters of the two-rowed determinant of (36)) is substituted, the combinants of (34) become the combinants of the two binary quintics which symmetrically represent the R^5 in space, and are actually invariants of this curve. The scheme of substitutions which must be made to transform the combinants of (3) and (4) into invariants of the R^5 in space whose parametric equations are (35), is obtained from (7) and is

$$(38) \quad \begin{aligned} \alpha &= |cdef|, & \beta &= |bdef|, & \gamma &= |bcef|, & \delta &= |bcd f|, & \lambda &= |adef|, \\ \alpha' &= |abcd|, & \beta' &= |abce|, & \gamma' &= |abde|, & \delta' &= |acde|, & \lambda' &= |abcf|, \\ \mu &= |acef|, & \mu' &= |abdf|, & \psi &= |bcde|, & \varphi &= |acdf|, & \chi &= |abef|. \end{aligned}$$

If these substitutions are made in the combinants of (3) and (4), the transformed expressions are invariants of (35). By the same transformation equations (9), (10), (19) and (20) become binary forms whose roots are sets of covariant points on (35). A very artificial algebraic interpretation could be given to all the combinants of (3) and (4) and to the sets of points represented by (9), (10), (19) and (20), but this would be mere pedantry. It is the purpose of this paper to form an algebraic foundation for analytic work, and geometrical interpretations are a secondary consideration. In order however to illustrate more fully the significance of the work, several applications which appear more or less on the surface are in place.

The combinant P transformed by (38) becomes the condition for the two quintics which symmetrically represent (35) to be apolar; each of necessity is self-apolar; these two quintics are apolar to all plane sections; hence P as an invariant of (35) is the condition for sets of the fundamental involution on (35) to be cut out of the R^5 (35) by planes on a line. The eliminant of (3) and (4) as an invariant of (35) is the condition for the R^5 to have a pentatactic plane or a plane cutting in 5 consecutive parameters. The equation (9) by means of (38) becomes the equation whose roots are tetratactic planes of the R^5 . The discriminant of this octavic breaks up into two factors, one of which is the invariant just mentioned; the other factor is the transform of the 9-flex tangents of the R^5 in the plane, and its vanishing conditions the correspondent of the cusp in the plane for the R^5 in space.

The conditions for the R^5 of (35) to have a node may be imposed by making $a_i = f_i$ in (35), for then 0 and ∞ are two different parameters which yield the coördinates of the same point. This condition* makes

$$(39) \quad \begin{cases} \mu = \mu' = \lambda = \lambda' = \varphi = \chi = 0, \\ \alpha = \delta', \quad \alpha' = \delta, \quad \beta = \gamma', \quad \beta' = \gamma, \end{cases}$$

* The identities (8) hold after the transformation (38) is made.

in (38). Let us call the I_3 of equation (20) G and 12 times the I_2 of (20) N . Also, let

$$(40) \quad 12K = Q - P^2$$

and

$$(41) \quad 3J = K^2 - N.$$

The conditions for (35) to have a node may be put in the form

$$(42) \quad K^3 - 72KJ + 216G = 0;$$

for (42) is satisfied when the conditions (39) hold, but is not true in general.

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